ANALYSIS-I

M.Sc., MATHEMATICS First Year

Semester – I, Paper-II

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M.Sc., MATHEMATICS – ANALSIS -I

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FOREWORD

Since its establishment in 1976, Acharya Nagarjuna University has been forging ahead in the path of progress and dynamism, offering a variety of courses and research contributions. I am extremely happy that by gaining 'A^{+}' grade from the NAAC in the year 2024, Acharya Nagarjuna University is offering educational opportunities at the UG, PG levels apart from research degrees to students from over 221 affiliated colleges spread over the two districts of Guntur and Prakasam.

The University has also started the Centre for Distance Education in 2003-04 with the aim of taking higher education to the doorstep of all the sectors of the society. The centre will be a great help to those who cannot join in colleges, those who cannot afford the exorbitant fees as regular students, and even to housewives desirous of pursuing higher studies. Acharya Nagarjuna University has started offering B.Sc., B.A., B.B.A., and B.Com courses at the Degree level and M.A., M.Com., M.Sc., M.B.A., and L.L.M., courses at the PG level from the academic year 2003-2004 onwards.

To facilitate easier understanding by students studying through the distance mode, these self-instruction materials have been prepared by eminent and experienced teachers. The lessons have been drafted with great care and expertise in the stipulated time by these teachers. Constructive ideas and scholarly suggestions are welcome from students and teachers involved respectively. Such ideas will be incorporated for the greater efficacy of this distance mode of education. For clarification of doubts and feedback, weekly classes and contact classes will be arranged at the UG and PG levels respectively.

It is my aim that students getting higher education through the Centre for Distance Education should improve their qualification, have better employment opportunities and in turn be part of country's progress. It is my fond desire that in the years to come, the Centre for Distance Education will go from strength to strength in the form of new courses and by catering to larger number of people. My congratulations to all the Directors, Academic Coordinators, Editors and Lesson-writers of the Centre who have helped in these endeavors.

Prof.K.GangadharaRao
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M.Sc. – MATHEMATICS SYLLABUS

SEMESTER – I

102MA24 :: ANALYSIS – I

Unit-I : **Numerical Sequences and Series**: Convergent sequences, Subsequence's, Cauchy Sequences. (3.1 to 3.14 of Chapter 3 of the Text book) (Questions not to be given in 3.1 to 3.14) Upper and Lower limits, Some special sequences, Series, Series of non-negative terms, Number Series, The Root and Ratio tests, Power series, Summation by parts, Absolute convergence, Addition and Multiplication of series. (3.15 to 3.51 of Chapter 3 of the Text book)

Unit-II: **Continuity**: Limits of functions, Continuous functions, Continuity and Compactness, Continuity and Connectedness. Discontinuities, Monotonic functions, Infinite limits and limits at infinity. (Chapter 4 of the Text book)

Unit-III : **Differentiation**: Derivative of a real function, Mean value theorems, The continuity of derivatives, L'Hospital's rule, Derivatives of higher order, Taylor's theorem. (5.1 to 5.15 of Chapter 5 of the Text book).

Unit-IV : Differentiation of vector-valued functions. **Riemann-Stieltjes Integral**: Definition and Existence of the Integral. (5.16 to 5.19 of Chapter 5 and 6.1 to 6.11 of Chapter 6 of the Text book)

Unit-V : Properties of the Integral, Integration and Differentiation, Integration of vector-valued functions, Rectifiable curves. (6.12 to 6.27 of Chapter 6 of the Text book)

CODE: 102MA24

M.Sc DEGREE EXAMINATION
First Semester
Mathematics::Paper II - ANALYSIS-1

MODEL QUESTION PAPER

Time : Three hours **Maximum : 70 marks**

Answer ONE question from each Unit. **(5 x 14 = 70)**

UNIT-I

1. Prove the following:
   - **(i)** If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$.
   - **(ii)** If $p > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$.
   - **(iii)** $\lim_{n\to\infty} \sqrt[n]{n} = 1$.
   - **(iv)** If $p > 0$ and α is real, then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$.
   - **(v)** If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

(OR)

2. A) Prove that $\sum_{n=2}^{\infty} \frac{1}{n(\log_e n)^p}$ converges if $p > 1$, and diverges if $p \leq 1$

   B) State and Prove Merten's Theorem.

UNIT – II

3. A) A mapping f of a metric space (X, d_1) into a metric space (Y, d_2) is continuous on X if and only if. $f^{-1}(V)$ is open in X for every open set V in Y.

   B) Suppose f *is* a continuous mapping of a compact metric space X into a metric space Y. Then $f(X)$ is compact.

(OR)

4. Let E be a non-compact set in $\mathbb{R}$. Then
   - **a)** There exists a continuous function on E which is not bounded.
   - **b)** There exists a continuous and bounded function on E which has no maximum. If, in addition, E is bounded, then
   - **c)** There exists a continuous function on E which is not uniformly continuous.

UNIT – III

5. A) Define $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, \text{if } (x \neq 0) \\ 0 \quad \text{if} \quad x = 0 \end{cases}$

   Then prove that f is continuous and differentiable at $x = 0$. Is f' continuous at

$x = 0$.

B) Let f be a real value function defined on $[a, b]$. If f has a local maximum at a point $x \in (a, b)$ and if $f'(x)$ exists, then $f'(x) = 0$.

(OR)

6. A) If f and g are continuous real functions on $[a, b]$ which are differentiable in (a, b), then there is a point $x \in (a, b)$ at which

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x).$$

B) State and Prove L – Hospital's rule theorem.

UNIT – IV

7. A) Suppose that f is a continuous mapping of $[a, b]$ into $\mathbb{R}^k$ and f is differentiable in (a, b). Then there exists $x \in (a, b)$ such that

$$|f(b) - f(a)| \leq (b - a)|f'(x)|.$$

B) Suppose f is defined in a neighbourhood of x, and suppose $f''(x)$ exists.

Show that $\lim_{h \to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = f''(x)$.

(OR)

8. A) $f \in R(\alpha)$ on $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$.

B) Suppose $f \in R(\alpha)$ on $[a, b]$, $m \leq f(x) \leq M$ for all $x \in [a, b]$, φ is continuous on $[m, M]$ and $h(x) = \varphi(f(x))$on $[a, b]$. Then $h \in R(\alpha)$ on $[a, b]$.

UNIT – V

9. Assume α increases monotonically on $[a, b]$ and $\alpha' \in R$ on $[a, b]$. Let f be a bounded real function defined on $[a, b]$. Then $f \in R(\alpha)$ on $[a, b]$ if and only if $f\alpha' \in R$ on $[a, b]$. Also $\int_a^b f \, d\alpha = \int_a^b f(x) \, \alpha'(x) dx$.

(OR)

10. A) State and Prove Fundamental Theorem of Calculus.

B) Suppose F and G are differentiable functions on $[a, b]$ $F' = f \in R$ and $G' = g \in R$ then $\int_a^b F(x)g(x) \, dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx$.

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CONTENTS

LESSON-1

NUMERICAL SEQUENCES

OBJECTIVES:

The objective of the lesson is to help the learners to understand

- ❖ To understand the concept of numerical sequences and their properties.
- ❖ To apply various types of sequences in mathematical and real world contexts.

STRUCTURE:

1.1 SEQUENCES:

A sequence in R is a function from N(the set of positive integers) into R. If s is a sequence, then the image $s(n)$ of $n \in N$ is usually denoted by s_n. It is customary to denote the sequence s by the symbol $\{s_n\}$. The image s_n of n is called the n^{th} term of sequence.
If s and t are two sequences in R, then t is said to be a sub-sequence of s if there exists a mapping $\varphi: N \to N$ such that (i) $t = s \circ \varphi$(ii) for each $n \in N$, there exists $m \in N$ such that $\varphi(i) \geq n$ for every $i \geq m$ in N. In other words, if $\{s_n\}$ is a sequence in R and $\{i_n\}$ is a sequence in N such that $i_1 < i_2 < \cdots < i_n < \cdots$, then $\{s_{i_n}\}$ is called a subsequence of $\{s_n\}$.
For example, if $s = \left\{\frac{1}{n}\right\}$ is a sequence in R, then $t = \left\{\frac{1}{2n-1}\right\}$ is a subsequence of s.

1.1.1 Definition: A function f defined on the set of all positive integers or the set of all non-negative integers is called as sequence.

1.1.2 Notation: If $f(n) = x_n$ for any positive integer n, we denote the sequence f by $\{x_n\}_{n\geq 1}$ or $\{x_1, x_2, \dots\dots, x_n, \dots .\}$.

1.1.3 Definition: Let (X, d) be a metric space. A sequence $\{x_n\}$ in X is said to be a convergent sequence if there is $x \in X$ such that for any $\varepsilon > 0$, there is a positive integer N such that $d(x_n, x) < \varepsilon \ \forall n \geq N$. Here x is called the limit of the sequence $\{x_n\}$, and we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

1.1.4 Definition: If the sequence $\{x_n\}$ is not convergent then it is said to be divergent.

1.1.5 Theorem: Let $\{x_n\}$ be a sequence in metric space X.
(i) $\{x_n\}$ converges to $p \in X$ if and only if every neighbourhood p contains all but finitely many terms of $\{x_n\}$

(ii) If $p \in X$ and $p' \in X$ and if $\{x_n\}$ converges to p and $\{x_n\}$ p' then $p = p'$.
(iii) If $\{x_n\}$ converges, then $\{x_n\}$ is bounded
(iv) If E containing $X(E \subseteq X)$ and if p is limit point of E, then there is a sequence $\{x_n\}$ in E such that $p = \lim_{n\to\infty} x_n$.

Proof: (ii) Given $\epsilon > 0$, choose "+ve" integer N_1, and N_2 such that ;

$d(x_n,x) < \frac{\epsilon}{2}, n \geq N_1$

$d(x_i,x) < \frac{\epsilon}{2}, n \geq N_2$. For $n \geq \max(N_1,N_2)$,

$0 \leq d(x,x') \leq d(x_{n'}x') + d(x_n,x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

Let $\epsilon > 0$, $d(x,x') = 0 \Rightarrow x = x'$

(iii) Choose a "+ve" integer $N \ni n > Nd(x_n,x) < 1$

If $M = \max\{1, d(x_1,x), \dots, d(x_N,x)\}$.

Then $d(x_N,x) \leq M \; \forall \; n \Rightarrow \{x_n\}$ is bounded

(iv) Given $\epsilon > 0$, choose "+ve" integer $N \ni \frac{1}{N} < \epsilon$ and

choose $x_n \in E \ni d(x_n,x) < \frac{1}{n}$. For $n > N$, $d(x_n,x) < \frac{1}{n} < \epsilon$ **i.e.,** $x_n \to x$.

1.1.6 Theorem: Suppose $\{s_n\},\{t_n\}$ are complex sequences, and the $\lim_{n\to\infty} s_n = s, \lim_{n\to\infty} t_n = t$. Then
(i) Then $\lim_{n\to\infty}(s_n + t_n) = s + t$
(ii) $\lim_{n\to\infty} cs_n = cs$, $\lim_{n\to\infty}(c + t_n) = c + s$ for any complex number c;
(iii) $\lim_{n\to\infty}(s_n . t_n) = st$
(iv) $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$, provided $s_n \neq 0$ $(n = 1,2, \dots \dots)$ and $s \neq 0$.

1.1.7 Definition: Let $\{P_n\}$ be a sequence in X. Let $\{n_k\}$ be a sequence of positive integers such that $n_1 < n_2 < \cdots \dots$ Then $\{P_{n_i}\}$ is called a subsequence of $\{P_n\}$.

If $\{P_{n_i}\}$ converges, then the limit of this sequence is called as a sub sequential limit of $\{P_n\}$.

1.1.8 Note: $\{P_n\}$ converges to P iff every sub sequence of $\{P_n\}$ converges to P.

1.1.9 Theorem:
(i) If $\{P_n\}$ is a sequence in a compact metric space in X then some sub sequence of $\{P_n\}$ converges to a point of X.
(ii) Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.

1.1.10 Theorem: The sub sequential limits of a sequence $\{P_n\}$ in a metric space X forms a closed subset of X.

1.1.11 Definition: A sequence $\{x_n\}$ in a metric space (X, d) is said to be Cauchy sequence if for every for every $\varepsilon > 0$, there is a positive integer N such that $d(x_m, x_n) < \varepsilon$ $\forall\, m, n \geq N$.

1.1.12 Definition: Let E be a subset of a metric space (X, d). Then the supremum of the set $\{d(x, y) | x, y \in E\}$ is called the diameter of E, and is denoted by $'diam\ E'$.

1.1.13 Theorem:

(i) In any metric space X, every convergent sequence is a Cauchy sequence.

(ii) If X is a compact metric space and if $\{P_n\}$ is a Cauchy sequence in X, then $\{P_n\}$ converges to same point of X.

(iii) In $\mathbb{R}^k$, every Cauchy sequence is convergent.

1.1.14 Definition: A metric space X is said to be a complete metric space if every Cauchy sequence in X is convergent.

1.1.15 Example:

1. The metric space $\mathbb{R}^k$ is a complete metric space.
2. Every compact metric space is complete.

1.1.16 Definition: A sequence $\{s_n\}$ of real numbers is said to be

(i) Monotonically increasing if $s_n \leq s_{n+1}$ for $n = 1,2,3, \ldots ....$

(ii) Monotonically increasing if $s_n \geq s_{n+1}$ for $n = 1,2,3, \ldots ....$

1.1.17 Definition: A sequence $\{s_n\}$ of real numbers is said to be a monotonic sequence if either $\{s_n\}$ is monotonically increasing (or) monotonically decreasing.

1.1.18 Note: Suppose $\{s_n\}$ is a monotonic sequence. Then $\{s_n\}$ converges if and only if it is bounded.

1.2 UPPER AND LOWER LIMITS:

1.2.1 Definition: Let $\{s_n\}$ be a sequence of real numbers.

(i) If for every real M there is an integer N such that $s_n \geq M$ $\forall\, n \geq N$, then we write $s_n \to \infty$ as $n \to \infty$

(ii) If for every real M there is an integer N such that $s_n \leq M$ $\forall\, n \geq N$, then we write $s_n \to -\infty$ as $n \to \infty$.

1.2.2 Definition: Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of all numbers $'x'$ in the extended real number system such that $s_{n_k} \to x$ for some sub sequence $\{s_{n_k}\}$ of $\{s_n\}$.

i.e., $E = \{x \in \mathbb{R}^\infty |\ \exists \text{ a subsequence } \{s_{n_k}\} \text{ of } \{s_n\} \ni\ s_{n_k} \to x\}$.

Then E contains all sub sequential limits of $\{s_n\}$ plus possibly the numbers $"+\infty", "-\infty"$. Define $s^* = \sup E$ and $s_* = \inf E$.

The numbers s^* and s_* are called upper limit and lower of the sequence $\{s_n\}$, respectively and we write $\lim_{n\to\infty} \sup s_n = s^*$ and $\lim_{n\to\infty} \sup s_n = s_*$.

1.2.3 Theorem: Let $\{s_n\}$ be a sequence of real numbers. Let $E = \{x \in \mathbb{R}^\infty |\ \exists \text{ a subsequence } \{s_{n_k}\} \text{ of } \{s_n\} \ni\ s_{n_k} \to x\}$. and $s^* = \sup E$. Then s^* has the following two properties:

(i) $s^* \in E$

(ii) If $x > s^*$, then there is an integer N such that $s_n < x\ \forall\ n \geq N$.

Moreover, s^* is the only number with the properties (i) and (ii).

1.2.4 Theorem: Let $\{s_n\}$ be a sequence of real numbers. Let $E = \{x \in \mathbb{R}^\infty |\ \exists \text{ a subsequence } \{s_{n_k}\} \text{ of } \{s_n\} \ni\ s_{n_k} \to x\}$. and $s_* = Inf E$. Then s^* has the following two properties:

(i) $s_* \in E$

(ii) If $x < s_*$, then there is an integer N such that $s_n > x\ \forall\ n \geq N$.

Moreover, s_* is the only number with the properties (i) and (ii).

1.2.5 Remark: Let $\{s_n\}$ and $\{t_n\}$ be two sequences of real numbers.

(1) If for fixed integer N, $s_n \leq t_n\ \forall\ n \geq N$, then

(i) $\lim_{n\to\infty} \inf s_n \leq \lim_{n\to\infty} \inf t_n$

(ii) $\lim_{n\to\infty} \sup s_n \leq \lim_{n\to\infty} \sup t_n$

(2) If for fixed integer N, $0 < s_n < t_n\ \forall\ n > N$ and if $t_n \to 0$, then $s_n \to 0$.

1.2.6 Theorem:

(i) If $p > 0$, then $\lim\limits_{n\to\infty} \frac{1}{n^p} = 0$

(ii) If $p > 0$, then $\lim\limits_{n\to\infty} \sqrt[n]{p} - 1$

(iii) $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

(iv) If $p > 0$ and α is real, then $\lim\limits_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$

(v) If $|x| < 1$, then $\lim\limits_{n\to\infty} x^n = 0$.

Proof:

(i) Suppose that $p > 0$ is a real number. Choose $\epsilon > 0$

Take a Positive integer N such that $\frac{1}{N^p} < \epsilon$

For any integer $n \geq N$, $\frac{1}{n^p} < \frac{1}{N^p} < \epsilon \Longrightarrow \frac{1}{n^p} < \epsilon\ \forall\ n \geq N$

So $\left|\frac{1}{n^p} - 0\right| = \left|\frac{1}{n^p}\right| = \frac{1}{n^p} < \epsilon\ \forall\ n \geq N$

This shows that $\lim\limits_{n\to\infty} \frac{1}{n^p} = 0$.

(ii) Suppose that $p > 0$

Case(i): Suppose that $p > 1$

For any integer $n \geq 1$, write $x_n = \sqrt[n]{p} - 1 \ldots\ldots\ldots.(1)$

It is clear that $x_n > 0 \;\forall\; n \geq 1$

Also, $1 + x_n = \sqrt[n]{p}$ (by (1))

$(1 + x_n)^n = p \Longrightarrow 1 + nx_n \leq p \;\Longrightarrow x_n \leq \frac{p-1}{n} \;\forall\; n \geq 1$

Let $\epsilon > 0$. Choose a positive integer N such that $\frac{p-1}{N} < \epsilon$

For any integer $n \geq N$, $\frac{p-1}{n} \leq \frac{p-1}{N} < \varepsilon \;\Longrightarrow \frac{p-1}{n} < \epsilon \;\forall\; n \geq N$

Now $|x_n - 0\,| = |x_n\,| = x_n \leq \frac{p-1}{n} < \epsilon \;\;\forall\; n \geq N$

Therefore $\lim_{n\to\infty} x_n = 0$. That is $\lim_{n\to\infty} \left(\sqrt[n]{p} - 1\right) = 0$

$\Longrightarrow \lim_{n\to\infty} \left(-1 + \sqrt[n]{p}\right) = 1$

Hence $\lim_{n\to\infty} \sqrt[n]{p} = 1$.

Case (ii): Suppose $p = 1$

Now $\lim_{n\to\infty} \sqrt[n]{p} = \lim_{n\to\infty} \sqrt[n]{1} = 1$

Case (iii): Suppose $p < 1$ i.e $0 < p < 1$

$\Longrightarrow \frac{1}{p} > 1$ is a real number.

So by case (i), we get that $\lim_{n\to\infty} \sqrt[n]{\frac{1}{p}} = 1 \Longrightarrow \lim_{n\to\infty} \frac{1}{\sqrt[n]{p}} = 1$

$\longrightarrow \lim_{n\to\infty} \sqrt[n]{p} = 1$

(iii) For any integer $n \geq 1$, write $x_n = \sqrt[n]{n} - 1$……..(1)

It is clear that $x_n \geq 0 \;\forall\; n \geq 1$

Also from (1) $x_{n+1} = \sqrt[n]{n} \Longrightarrow n = (1 + x_n)^n \Longrightarrow n \geq \frac{n(n-1)}{2} x_n{}^2$

$$\Longrightarrow x_n{}^2 \leq \frac{2}{n-1} \Longrightarrow x_n \leq \sqrt{\frac{2}{n-1}} \;\forall\; n \geq 2$$

Choose $\epsilon > 0$. Take a positive integer N such that $\frac{2}{N-1} < \epsilon^2$

For any integer $n \geq N$, $|x_n - 0| = |x_n| = x_n < \sqrt{\frac{2}{n-1}} < \sqrt{\frac{2}{N-1}} < \epsilon$

Therefore $\lim_{n\to\infty} x_n = 0$.

$\Longrightarrow \lim_{n\to\infty} (\sqrt[n]{n} - 1) - 0 \Longrightarrow \lim_{n\to\infty} \sqrt[n]{p} - 1.$

(iv) Suppose that $p > 0$ and α are real numbers.

Let k be a positive integer such that $k > \alpha$

For any $n > 2k$,

$(1 + p)^n > n_{c_k} p^k = \frac{n!}{(n-k)!\,k!} p^k = \frac{n(n-1)\ldots.[n-(k-1)]}{k!} p^k$

$= \frac{n^k p^k}{2^k k!}$

$\Longrightarrow (1+p)^n > \frac{n^k p^k}{2^k k!} \Longrightarrow \frac{n^k}{(1+p)^n} < \frac{2^k k!}{p^k} \forall\, n > 2k \dots\dots (1)$

Let $\epsilon > 0$

Now $\lim_{n\to\infty} n^{\alpha-k} = \lim_{n\to\infty} \frac{1}{n^{\alpha-k}} = 0$

Therefore $\lim_{n\to\infty} \left(\frac{2^k k!}{p^k}\right) n^{\alpha-k} = 0$

$\Longrightarrow \exists$ a positive integer N such that $\left|\left(\frac{2^k k!}{p^k}\right) n^{\alpha-k} - 0\right| < \epsilon \;\forall\, n \geq N.$

Take $N_1 = \max\{N, 2k\}$

For every $\forall\, n \geq N_1$ $\left|\frac{n^\alpha}{(1+p)^n}\right| < \left|\frac{n^{\alpha-k} 2^k k!}{p^k}\right| < \epsilon$

$\Longrightarrow \left|\frac{n^\alpha}{(1+p)^n} - 0\right| < \epsilon$

This shows that $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$

(v) Suppose that x is a real number such that $|x| < 1 \Longrightarrow \frac{1}{|x|} > 1$

$\Longrightarrow \frac{1}{|x|} - 1 > 0$

Let $p = \frac{1}{|x|} - 1$

Then p is a real number such that $p > 0$

By (iv) $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$ for every real number α.

Taking $\alpha = 0$, we get that $\lim_{n\to\infty} \frac{1}{(1+p)^n} = 0$

$\Longrightarrow \lim_{n\to\infty} \frac{1}{(\frac{1}{|x|})^n} = 0 \Longrightarrow \lim_{n\to\infty} |x|^n = 0 \Longrightarrow \lim_{n\to\infty} x^n = 0.$

1.3 SOME MORE EXAMPLES WITH SOLUTIONS:

1.3.1 Example: Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. is the converse true?

Solution: Let $\epsilon > 0$.

Since the sequence $\{s_n\}$ is a Cauchy sequence, there exists N such that $|s_m - s_n| < \epsilon$ for all $m > N$ and $n > N$.

We then have $\big||s_m| - |s_n|\big| \leq |s_m - s_n| < \epsilon$ for all $m > N$ and $n > N$.

Hence the sequence $\{|s_n|\}$ is also a Cauchy sequence, and therefore must converge.

The converse is not true, as shown by the sequence $\{s_n\}$ with $s_n = (-1)^n$.

1.3.2 Example: Calculate $\lim_{n\to\infty} \left(\sqrt{n^2+n} - n\right)$.

Solution: Multiplying and dividing by $\sqrt{n^2+n} + n$ yields

$$\sqrt{n^2+n}-n=\frac{n}{\sqrt{n^2+n}+n}=\frac{1}{\sqrt{1+\frac{1}{n}}+1}$$

It follows that the limit is $\frac{1}{2}$.

1.3.3 Example: If $s_1=\sqrt{2}$ and $s_{n+1}=\sqrt{2+\sqrt{s_n}}$ $(n=1,2,3\,...)$, prove that $\{s_n\}$ converges, and that $s_n<2$ for $n=1,2,3\,....$

Solution: Since $\sqrt{2}<2$, it is manifest that if $s_n<2$, then $s_{n+1}<\sqrt{2+2}=2$.

Hence it follows by induction that $\sqrt{2}<s_n<2$ for all n.

In view of this fact, it follows that $(s_n-2)(s_n+1)<0$ for all $n>1$,

i.e., $s_n>{s_n}^2-2=s_{n-1}$.

Hence the sequence is an increasing sequence that is bounded above (by 2) and so converges.

Since the limit s satisfies $s^2-s-2=0$

It follows that the limit is2.

1.3.4 Example: Find the upper and lower limits of the sequence $\{s_n\}$ defined by $s_1=0$; $s_{2m}=\frac{s_{2m-1}}{2}$; $s_{2m+1}=\frac{1}{2}+s_{2m}$.

Solution: We shall prove by induction that

$s_{2m}=\frac{1}{2}-\frac{1}{2^m}$ and $s_{2m+1}=1-\frac{1}{2^m}$ for $m=1,2,....$

The second of these equalities is a direct consequence of the first, and so we need only prove the first.

Immediate computation shows that $s_2=0$ and $s_3=\frac{1}{2}$.

Hence assume that both formulas hold for $m\le r$.

Then $s_{r+2}=\frac{1}{2}s_{2r+1}=\frac{1}{2}\left(1-\frac{1}{2^r}\right)=\frac{1}{2}-\frac{1}{2^{r+1}}$.

This completes the induction.

We thus have $\lim_{n\to\infty} sup s_n=1$ and $\lim_{n\to\infty} inf s_n=\frac{1}{2}$.

1.3.5 Example: For any two real sequences $\{a_n\},\{b_n\}$ prove that $\lim_{n\to\infty} sup(a_n+b_n)\le\lim_{n\to\infty} sup\,a_n+\lim_{n\to\infty} sup\,b_n$, provided the sum on the right is not of the form $\infty-\infty$.

Solution: Since the case when $\lim_{n\to\infty} sup a_n = +\infty$ and $\lim_{n\to\infty} sup b_n = -\infty$ has been excluded from consideration.

we note that the inequality is obvious if $\lim_{n\to\infty} sup a_n = +\infty$.

Hence we shall assume that $\{a_n\}$ is bounded above.

Let $\{n_k\}$ be a subsequence of the positive integers such that
$\lim_{k\to\infty} sup(a_{n_k} + b_{n_k}) = \lim_{n\to\infty} sup(a_n + b_n)$.

Then choose a subsequence of the positive integers $\{k_m\}$ such that $\lim_{m\to\infty} a_{n_{k_m}} = \lim_{k\to\infty} sup\ a_{n_k}$

The subsequence $a_{n_{k_m}} + b_{n_{k_m}}$ still converges to the same limit as $a_{n_k} + b_{n_k}$,

i.e., to $\lim_{n\to\infty} sup(a_n + b_n)$.

Hence, since a_{n_k} is bounded above (so that $\lim_{k\to\infty} sup\ a_{n_k}$ is finite),

It follows that $b_{n_{k_m}}$ converges to the difference

$$\lim_{m\to\infty} b_{n_{k_m}} = \lim_{m\to\infty} \left(a_{n_{k_m}} + b_{n_{k_m}}\right) - \lim_{m\to\infty} a_{n_{k_m}}.$$

Thus we have proved that there exists sub sequences $\left\{a_{n_{k_m}}\right\}$ and $\left\{b_{n_{k_m}}\right\}$ which converge to limits a and b respectively such that $a + b = \lim_{n\to\infty} sup(a_n + b_n)$.

Since a is the limit of a subsequence of $\{a_n\}$ and b is the limit of a subsequence of $\{b_n\}$

It follows that a is the limit of a subsequence of $a \le \lim_{n\to\infty} sup a_n$ and $b \le \lim_{n\to\infty} sup\, b_n$, from which the desired inequality follows.

Exercises

1. Write a formula for s_n for each of the following sequences:

(i) $1, -1, 1, -1$

(ii) $2,1,4,3,6,5,8,7, \ldots..$

(iii) $1,3,6,10,15, \ldots.$

(iv) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{5}, \ldots$

Ans. (i) $s_n = 1$ if n is odd, $s_n = -1$ if n is even,(ii) $s_n = n + 1$ if n is odd, $s_n = n - 1$ if n is even.

(iii) $s_n = \frac{n(n+1)}{2}$ (iv) $s_n = \frac{n}{n+1}$

2. If l and m are real numbers such that $l \leq m + \epsilon$ for every $\epsilon > 0$, then prove that $l \leq m$.

3. Use the definition of the limit of a sequence to show that the limit of a sequence $\{s_n\}$ where $s_n = \frac{2n}{n+3}$ is 2.

4. If the sequence $\{s_n\}$ converge to l, then prove that the sequence $\{|s_n|\}$ converges to $\{|l|\}$
[**Hint:** Use the inequality $||x| - |y|| \leq |x - y|\ x, y \in R$.

5. Give an example of a sequence $\{s_n\}$ of real numbers such that $\{|s_n|\}$ converges but $\{s_n\}$ does not.

Ans: One such sequence is $\{s_n\}$ where $s_n = (-1)^n$.

6. Prove that the sequence $\{s_n\}$ where $s_n = \sqrt{n}$ diverges to ∞.

7. Is the sequence $\{\sin n\pi\}$ convergent?

Ans: Yes, it converges to 0.

8. If $\{s_n\}$ and $\{t_n\}$ are non-decreasing bounded sequences, and if $s_n \leq t_n (n \in N)$, prove that $\lim\ s_n \leq \lim\ t_n$.

9. Show that $\lim \frac{x^n}{n!} = 0$, where x is any number.

10. Show that $\lim n! \left(\frac{a}{n}\right)^n = 0$ or $+\infty$, according as $a < e$ or $a > e$.

11. If x_1, y_1 are positive and if for $n \geq 1$, $2x_{n+1} = x_n + y_n$ and $\frac{2}{y_{n+1}} = \frac{1}{x_n} + \frac{1}{y_n}$, Show that $\{x_n\}$ and $\{y_n\}$ are monotonic sequences and approach a common limit l, where $l^2 = x_1 y_1$.

12. If $a_n = \frac{a}{1+a_{n-1}}$, where a, a_n are positive, show that the sequence $\{a_n\}$ tends to definite limit l, the positive root of the equation $x^2 + x = a$.

13. If k is positive and $\alpha, -\beta$ are positive and negative roots of $x^2 - x - k = 0$,prove that if $v_n = k - v_{n-1}$ and $v_1 < k$ then $\lim v_n = \beta$.

14. If $0 < u_1 < u_2$ and $u_n = \frac{2u_{n-1}u_{n-2}}{u_{n-1} + u_{n-2}}$ 9i.e., u_n is the harmonic mean of u_{n-1} and u_{n-2}), show that $\lim u_n = \frac{1}{3} u_1 u_2 / (2u_1 + u_2)$.

15. If $\sigma_n = \left(\frac{1}{n}\right)(s_1 + s_2 + \cdots + s_n)(n \in N)$, prove that $\overline{\lim}\ \sigma_n < \overline{\lim}\ s_n$ and $\underline{\lim}\ \sigma_n \geq \underline{\lim}\ s_n$

16. If $\{s_n\}$ is a Cauchy sequence of real numbers which has sub-sequence converging to l, prove that $\{s_n\}$ itself converges to l.

1.4 SUMMARY:

This lesson is designed to introduce learners to the fundamental concept of numerical sequences, exploring their properties, and applying them to real-world contexts. This lesson provides a solid foundation for learners to develop their understanding of numerical sequences and their applications, preparing them for more advanced mathematical concepts and real-world problem-solving. Key Takaways of this lesson are Definitions and theorems of numerical sequences, Upper and lower limits of sequences, Applications of sequences in mathematics and real-world problems, and Examples and exercises to reinforce understanding.

1.5 TECHNICAL TERMS:

- Cauchy Sequence
- Compact Metric Space
- Complex Sequences
- Convergent
- Diameter
- Divergent
- Limit of the Sequence
- Metric space
- Monotonically Decreasing
- Monotonically Increasing
- Neighbourhood
- Sequence
- Subsequence
- Supremum
- Upper limit and lower limit

1.6 SELF ASSESSMENT QUESTIONS

1. Calculate $\lim_{n\to\infty}\left(\sqrt{n^2+n}-n\right)$.

2. Find the upper and lower limits of the sequence $\{s_n\}$ defined by $s_1 = 0$; $s_{2m} = \frac{s_{2m-1}}{2}$; $s_{2m+1} = \frac{1}{2} + s_{2m}$.

3. If l and m are real numbers such that $l \leq m + \epsilon$ for every $\epsilon > 0$, then prove that $l \leq m$.

4. If the sequence $\{s_n\}$ converge to l, then prove that the sequence $\{|s_n|\}$ converges to $\{|l|\}$

[**Hint:** Use the inequality $\big||x| - |y|\big| \leq |x - y|\ x, y \in R$.

5. Is the sequence $\{\sin n\pi\}$ convergent?

Ans: Yes, it converges to 0.

6. Give an example of a sequence $\{s_n\}$ of real numbers such that $\{|s_n|\}$ converges but $\{s_n\}$ does not.

Ans: One such sequence is $\{s_n\}$ where $s_n = (-1)^n$.

7. If $\{s_n\}$ and $\{t_n\}$ are non-decreasing bounded sequences, and if $s_n \leq t_n (n \in N)$, prove that $\lim s_n \leq \lim t_n$.

1.7 SUGGESTED READINGS:

1. Principles of Mathematical Analysis, Third Edition, Mc Graw-Hill International Editions Walter Rudin.

2. Mathematical Analysis by Tom M. Apostol, Narosa Publishing House, 2nd Edition, 1985.

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LESSON-2

NUMERICAL SERIES

OBJECTIVES:

The objective of the lesson is to help the learners to understand

- ❖ To understand the concept of numerical series including convergence and divergences.
- ❖ To apply series properties for solving mathematical Problems.

STRUCTURE:

2.1 SERIES:

An expression of the form $u_1 + u_2 + u_3 + \cdots + u_n + \cdots$, in which every term is followed by another according to some definite law is called a series. If the series contains a finite number of terms, it is called a finite series; in case the number of terms is unlimited, it is called an infinite series. The above series is symbolically denoted as $\sum_{n=1}^{\infty} u_n$.

If $R_n = u_{n+1} + u_{n+2} + \cdots$, then R_n is called remainder after n terms of the series.

Dependence of series on sequences: If $S_n = u_1 + u_2 + u_3 + \cdots + u_n$, then S_n is called the sum to n terms or the n^{th} partial sum of the series $\sum_{n=1}^{\infty} u_n$.

Thus we can express the series $\sum u_n$ as a sequence of the partial sums $\{S_n\}$. In other words, the behaviour of the series $\sum u_n$ is the same as the behaviour of the sequence S_1, $S_2, S_n, \ldots$

2.1.1 Definition : Given a sequence $\{x_n\}$, we define

$S_n - \sum u_k - u_1 + u_2 + u_3 + \cdots + u_n$ If $\{S_n\}$ converges, say to S, we write;

$\sum a_k = S = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$ and call $\sum a_n$, an infinite series. We say $\sum a_n$ diverges if it doesn't converges.

2.1.2 Theorem: $\sum a_n$ converges if and only if given $\epsilon > 0$, $\exists$ a positive integer

$N \ni n \geq m \geq N \Longrightarrow |\sum a_k| < \epsilon,\ k = 1,2, \ldots, n$

Proof: Notice $S_n - S_m = \sum a_k$ and apply Cauchy criterion to S_n.

2.1.3 Theorem: If $\sum a_n < \infty$, $a_n \to 0$

Proof: Let $s = \sum a_k \cdot S_n \to s, S_{n+1} \to s$

So $a_{n+1} = S_{n+1} - S_n \to 0$.

2.1.4 Remark: (1) If $a_n \to 0$, $\sum a_n$ need not converges consider $\sum \frac{1}{n}\frac{1}{n} \to 0$. If

$S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$, and $\{S_n\}$ converges, $\exists$ positive integer $|S_{2N} - S_0| < \frac{1}{2}$

But $S_{2N} - S_0 = \left(\frac{1}{N+1}\right) + \left(\frac{1}{N+2}\right) + \cdots + \left(\frac{1}{N+N}\right) > \left(\frac{N}{N+N}\right) = \frac{1}{2}$

So $\sum a_n$ doesn't converge.

2.1.5 Theorem: Let $\sum a_n \geq 0$. Then $\sum a_n$ converges if and only if S_n is bounded above.

Proof: $S_n - S_{n-1} = a_n > 0$ so that $\{S_n\}$ is $\uparrow$ and $\{S_n\}$ converges

$\Leftrightarrow \{S_n\}$ is bounded above.

2.1.6 Theorem: If $|a_n| \leq c_n$ for $n \geq N$, and if $\sum c_n$ converges, then $\sum a_n$ converges.

If $a_n \geq d_n \geq 0$ for $n \geq N$, and if $\sum d_n$ diverges, then $\sum a_n$ diverges.

Proof: Given $\epsilon > 0$, $\exists$ a positive integer $N \; \forall \; x \geq m \geq N \; \exists \; C_k < \epsilon$

So $|\sum a_k| \leq \sum |a_n| \leq \sum c_k < \epsilon$

So that (1) follows.

If $\sum a_n$ converges, $\sum d_n$ converges by (1).

2.1.7 Theorem: If $0 \leq x < 1$, $\sum X^x = \frac{1}{(1-X)} \cdot$ If $x \geq 1$, $\sum X^x$ diverges.

Proof: for $x \neq 1$, $S_x - \sum X^x - 1 + x + x^2 + \cdots + x^x$

$= \frac{(1-x^{n+1})}{(1-x)} \to \frac{1}{(1-x)} \cdot$ Since $x^x \to 0$ for $|x| < 1$

If $x > 1$, $S_n > n \to \infty \cdot$ So $\sum X^x$ diverges for $x > 1$

We call $\sum X^x (0 \leq x < 1)$, the Geometric series.

2.1.8 Theorem: (CAUCHY CONDENSATION TEST)

Let $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. Then $\sum a_n$ Converges, if and only if $\sum 2^k a_{2_k}$ converges.

Proof: Let $S_n = \sum a_k$ and $t_k = \sum 2^j a_{2j}$ for $n < k$,

$S_n = a_1 + (a_2 + a_3) + \cdots + (a_{2k} + \cdots + a_{2k+1})$

$\geq a_1 + 2a_1 + \cdots + 2^k a_{2k} = t_k$ (use $a_n \downarrow$)

i.e., $S_n > t_k$, $S_n = t_k$ ....(1)

For $n > k$,

$S_n \geq a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2k-1} + \cdots + a_{2k})$

$\geq \frac{1}{2}(a_1 + a_2 + 2a_4 + \cdots + 2^{k-1} a_{2k})$

$\Rightarrow 2S_n \geq a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2k}$

$= t_k$

i.e., $2S_n \geq t_k$ ....(2)

From (1) & (2); So $\{S_n\}$ is bounded $\Leftrightarrow \{t_k\}$ is bounded.

The Theorem follows.

2.1.9 Theorem : $\sum \frac{1}{n^p}$ converges if and only if $p > 1$.

Proof: If $p \leq 1$, $\frac{1}{n^p} \geq \frac{1}{n}$ and $\sum \frac{1}{n}$ diverges.

So $\sum \frac{1}{n^p}$ diverges for $p \leq 1$

If $p > 1$, $\sum \frac{1}{n^p}$ is $\downarrow$ and $\sum \frac{1}{n^p}$ converges if $\sum 2^k \left(\frac{1}{2^{kp}}\right)$

$\sum 2^{k(1-p)}$ converges.

Now $2^{1-p} < 1 \Leftrightarrow 1 - p < 0$

$\Leftrightarrow p > 1$.

2.1.10 Theorem : $\sum_{n=2}^{\infty} \frac{1}{n(\log_e n)^p}$ converges if $p > 1$, and diverges if $p \leq 1$..

Proof: Clearly, sequence $\{\log_e n\}$ is monotonically increasing sequence of positive terms for $n \geq 2$.

The sequence $\{n \log n\}$ is monotonically increasing sequence of positive terms.

The sequence $\left\{\frac{1}{n \log n}\right\}$ is monotonically decreasing sequence of positive terms.

So by known theorem, we have that $\sum_{n=2}^{\infty} \frac{1}{n(\log_e n)^p}$ converges if and only if

$\sum_{k-1}^{\infty} 2^k \frac{1}{2^k(\log_2 k)^p}$ converges.

$\Leftrightarrow \sum_{k=1}^{\infty} \frac{1}{(k \log 2)^p}$ converges.

$\Leftrightarrow \sum_{k=1}^{\infty} \frac{1}{k^p(\log 2)^p}$ converges

$\Leftrightarrow \frac{1}{(\log 2)^p} \sum_{k=1}^{\infty} \frac{1}{k^p}$ converges.........(1)

Therefore $\sum_{n=2}^{\infty} \frac{1}{n(\log_e n)^p}$ converges $\Leftrightarrow \frac{1}{(\log 2)^p} \sum_{k=1}^{\infty} \frac{1}{k^p}$ converges.

By the above theorem (1.3.9) the series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$.

$\Rightarrow \frac{1}{(\log 2)^p} \sum_{k=1}^{\infty} \frac{1}{k^p}$ converges when $p > 1$.

So by (1) the series $\sum_{n=2}^{\infty} \frac{1}{n(\log_e n)^p}$ converges if $p > 1$

Again by the same theorem the series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges if $p \leq 1$.

$\Rightarrow \frac{1}{(\log 2)^p} \sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges if $p \leq 1$

So by (1) the series $\sum_{n=2}^{\infty} \frac{1}{n(\log_e n)^p}$ diverges if $p \leq 1$.

2.1.11 Note: The series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.

Proof: For each integer $n \geq 1$, write $s_n = \sum_{k=0}^{n} \frac{1}{k!}$

Now for $n \geq 1$, $s_n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$

$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} < 3$

$\Rightarrow s_n < 3 \ \forall \ n$

Therefore the sequence $\{s_n\}$ of partial sums is bounded and also monotonically increasing sequence.

Hence by known theorem $\{s_n\}$ is convergent.

So by definition, the series $\sum_{n=0}^{\infty}\frac{1}{n!}$ converges.

2.1.12 Theorem: $\lim_{x\to\infty}\left(1+\frac{1}{n}\right)^n = \sum\frac{1}{n!}$.

Proof: $S_n = \sum\frac{1}{n!}$ and $t_k = \left(1+\frac{1}{x}\right)^n$

Notice $\lim\limits_{x\to\infty} S_n$ exists. S_n is ↑ and;

$$S_n = 1+1+\frac{1}{2}!+\frac{1}{3}!+\cdots+\frac{1}{n}!$$

$$< 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}} < 3$$

By the Binomial Theorem.

$$t_n = 1+1+\frac{1}{2}!\left(1-\frac{1}{n}\right)+\frac{1}{3}!\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots$$
$$+\frac{1}{n}!\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(2-\frac{(n-1)}{n}\right)$$

$\Rightarrow t_n \le \sum\frac{1}{n!}$ ....(1)

Again for $n \ge N$, t_n is and so;

$$t_n \ge 1+1+\frac{1}{2}!\left(1-\frac{1}{n}\right)+\cdots+\frac{1}{N}!\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{(N-1)}{n}\right)$$

Fix N, and let $n\to\infty$

So $\lim t_n \ge 1+1+\frac{1}{2}!+\cdots+\frac{1}{N}! = S_n$

Let $\lim\limits_{x\to\infty} t_n \ge \sum\frac{1}{n!}$ ....(2)

By (1) & (2), $\lim t_n = e$ and the theorem follows.

2.1.13 Definition: we define $e = \sum_{n=0}^{\infty}\frac{1}{n!}$

2.1.14 Theorem : e is irrational

Proof: If S_n is the n[th] partial sum of $\frac{1}{n!}$, then;

$$0 < e - S_n - \frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\frac{1}{(n+3)!}+\cdots$$

$$< \frac{1}{(n+1)!}\left[1+\frac{1}{(n+1)}+\frac{1}{(n+1)^2}+\cdots\right] = \frac{1}{n}n! \text{ ....(1)}$$

If e were rational, say $e = \frac{p}{q}, p = q \in N$

Then $0 < e - S_q < \frac{1}{q}q!$ By (1)

$\Rightarrow 0 < q!\left(e - S_q\right) < \frac{1}{q} \le 1$

Now $q!\,e$ and $q!\ S_q$ are integers so that $q!\left(e - S_q\right)$ is an integer between 0 & 1. This is impossible, so e must be irrational.

2.1.15 Theorem: [Root test] state and prove Root test (or)

Given $\sum_{n=1}^{\infty} a_n$; put $\alpha = \lim_{n\to\infty}\sup|a_n|^{\frac{1}{n}}$. Then

(i) If $\alpha < 1$, $\sum a_n$ converges;

(ii) If $\alpha > 1$, $\sum a_n$ diverges;

(iii) If $\alpha = 1$ the test gives no information.

Proof: Let $\alpha = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}$

(i) Suppose $\alpha < 1$

Let β be a real number such that $\alpha < \beta < 1$

Since $0 < \beta < 1$, [by theorem 1.3.8], we know that the series $\sum_{i=1}^{n} \beta^n$ is convergent also since $\beta > \alpha$

$= \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}$ there exists a "+ve" integer N such that $\sqrt[n]{|a_n|} < \beta, \forall\, n \geq N$.

$\Rightarrow |a_n| < \beta^n, \forall\, n \geq N$

Since $\sum_{i=1}^{n} \beta^n$ is convergent, by composition test

$\sum_{n=1}^{\infty} a_n$ is convergent

(ii) $\alpha > 1$

Since $\alpha = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|} > 1$

We know that there is a sequence $\{n_n\}$ of "+ve" integers such that $\sqrt[n_k]{|a_{n_k}|} \to \alpha > 1$

As $\alpha > 1$ there are infinitely many "+ve" integer $n_k \ni |a_{n_k}| > 1$

$\Rightarrow$ there are infinitely many "+ve" integer $n \ni |a_n| > 1$.........(1)

Now we have to prove that

$\sum_{n=1}^{\infty} a_n$ is diverges

on the contrary suppose that

$\sum_{n=1}^{\infty} a_n$ is converges

$\Rightarrow \lim_{n\to\infty} a_n = 0.$

So, for $\epsilon = 1,\ \exists$ a positive integer $N \ni |a_n| = |a_n - 0| < 1\ \forall\, n \geq N$

which is contradiction to (1)

Therefore, $\sum_{n=1}^{\infty} a_n$ is diverges.

(iii) Consider the series

$\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Now $\alpha = \lim \sup \sqrt[n]{\left|\frac{1}{n}\right|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{n}\right|} \sup \frac{1}{\sqrt[n]{n}}$

$$= \frac{1}{\lim_{n\to\infty} \sqrt[n]{n}} = \frac{1}{1} = 1$$

But we know that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent

Also $\alpha = \lim_{n\to\infty} \sup \sqrt[n]{\left|\frac{1}{n^2}\right|}$ (by theorem 1.3.9)

$$= \lim_{n\to\infty} \sup \sqrt[n]{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \sup \left(\frac{1}{\frac{1}{n^n}}\right)^2 = \lim_{n\to\infty} \left(\frac{1}{\frac{1}{n^n}}\right)^2 = \frac{1}{1} = 1$$

But $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent [by theorem 1.3.9]

So if $\alpha = 1$, the test gives no information.

2.1.16 Theorem: (The Ratio Test) Let $a_n \neq 0$ then $\sum a_n$

(1) Converges if $\lim \sup \left[\left|\frac{(a_n+1)}{a_n}\right|\right] < 1$ and

(2) Diverges if $\lim \left[\left|\frac{(a_n+1)}{a_n}\right|\right] > 1$.

Proof: The given series is $\sum_{n=1}^{\infty} a_n$

(1) Suppose that $\lim_{n\to\infty} \sup \left|\frac{a_n+1}{a_n}\right| < 1$

Then there exists a real number β such that $\lim_{n\to\infty} \sup \left|\frac{a_n+1}{a_n}\right| < \beta < 1$

$\Rightarrow$ $\exists$ a "+ve" integer $N \ni \left|\frac{a_n+1}{a_n}\right| < \beta\ \forall\ n \geq N$.

$\Rightarrow |a_n + 1| < \beta\ |a_n|, \forall\ n \geq N$

For any "+ve" integer p,

$|a_N + 1| < \beta\ |a_N|$

$|a_N + 2| < \beta|a_N + 1| < \beta \cdot \beta\ |a_N| = \beta^2\ |a_N|$

$|a_N + p| < \beta^p\ |a_N|$

For any $n \geq N$, $|a_n| = |a_{N+n-N}| < \beta^{n-N}|a_N|$

$= \beta^n \beta^{-N} |a_N|$

As $0 < \beta < 1$,

We know that

$\sum_{n=1}^{\infty} \beta^n$ converges

$\Rightarrow \beta^{-N} |a_N| \sum_{n=1}^{\infty} \beta^n$ converges

$\Rightarrow \sum_{n=1}^{\infty} |a_N|\ \beta^{-N} \beta^n$ converges

Since for $n \geq N$, $|a_n| \leq |a_N|\beta^{-N}\beta^n$

By comparison test the series

$\sum_{n=1}^{\infty} a_n$ converges

(2) Suppose that $\left|\frac{a_n+1}{a_n}\right| \geq$ for $n > n_0$, where n_0 is some fixed "+ve" integer

Then $|a_n + 1| < |a_n|$ for all $n \geq n_0$

we have to prove that series $\sum_{n=1}^{\infty} a_n$ diverges

on contrary way suppose that $\sum_{n=1}^{\infty} a_n$ is converges

by a know theorem $\lim_{n\to\infty} a_n = 0$

therefore $\epsilon = |a_{n_0}| > 0\ \exists$ there is a "+ve" integer

$N_1 \ni |a_n| < \epsilon = |a_{n_0}|\ \forall\ n \geq N_1$

Taken $N = max\{n_0, N_1\} + 1$

Then $|a_N| < |a_{n_0}|$ ....(1)

Since $N > n_0$, $|a_N| = |a_{n_0} + N - n_0| \geq |a_{n_0}|$

which is a contradiction to (1)

Therefore, $\sum_{n=1}^{\infty} a_n$ diverges.

2.1.17 Theorem: For any sequence $\{c_n\}$ of positive numbers,

(i) $\lim_{n\to\infty} \inf \frac{c_{n+1}}{c_n} < \lim_{n\to\infty} \inf \sqrt[n]{c_n}$ and

(ii) $\lim_{n\to\infty} \sup \sqrt[n]{c_n} \leq \lim_{n\to\infty} \sup \frac{c_{n+1}}{c_n}$

Proof: Let $\{c_n\}$ be a sequence of positive real numbers

(i) Let $\alpha = \lim_{n\to\infty} \inf \frac{c_{n+1}}{c_n}$

If $\alpha = -\infty$ there is nothing to prove

Suppose $\alpha \neq -\infty$ i.e, $-\infty < \alpha$

Choose a number β such that $-\infty < \beta < \alpha$

So $\exists$ a positive integer $N \ni \frac{c_{n+1}}{c_n} \geq \beta \; \forall \, n \geq N$

$\Rightarrow c_{n+1} \geq \beta c_n \quad \forall \, n \geq N$

In particular for any number $p > 0$, $c_{N+1} \geq \beta c_N$

$c_{N+2} \geq \beta c_{N+1} \geq \beta^2 c_N$ and so on we get $c_{N+p} \geq \beta^p c_N$

So for any integer $n \geq N$, $c_n = c_{n-N+N} = c_{N+(n-N)} \geq \beta^{n-N} c_N$

$\Rightarrow c_n \geq \beta^n \beta^{-N} c_N \;\; \forall \, n \geq N$

$\sqrt[n]{c_n} \geq \sqrt[n]{\beta^{-N}} \; c_N \beta \;\; \forall \, n \geq N$

$\Rightarrow \lim\limits_{n\to\infty} \inf \sqrt[n]{c_n} \geq \beta \lim\limits_{n\to\infty} \inf \sqrt[n]{\beta^{-N} c_N} = \beta(1)$

Since $\beta < \alpha$ is arbitrary, we have that $\lim_{n\to\infty} \inf \sqrt[n]{c_n} \geq \alpha$

$\lim_{n\to\infty} \inf \sqrt[n]{c_n} \geq \lim_{n\to\infty} \inf \frac{c_{n+1}}{c_n}$

(ii) Let $\alpha = \lim_{n\to\infty} \sup \frac{c_{n+1}}{c_n}$

If $\alpha = +\infty$ there is nothing to prove

Suppose α is a finite real number

Choose a number β such that $\alpha < \beta$

So $\exists$ a positive integer $N \ni \frac{c_{n+1}}{c_n} \leq \beta \; \forall \, n \geq N$

$\Rightarrow c_{n+1} \leq \beta c_n \quad \forall \, n \geq N$

In particular for any number $p > 0$, $c_{N+1} \leq \beta c_N$

$c_{N+2} \leq \beta c_{N+1} \leq \beta^2 c_N$ and so on we get $c_{N+p} \leq \beta^p c_N$

So for any integer $n \geq N$, $c_n = c_{n-N+N} = c_{N+(n-N)} \leq \beta^{n-N} c_N$

$\Rightarrow c_n \leq \beta^n \beta^{-N} c_N \;\; \forall \, n \geq N$

$\sqrt[n]{c_n} \leq \sqrt[n]{\beta^{-N}}\, c_N \beta \quad \forall\, n \geq N$

$\Rightarrow \lim_{n\to\infty} \sup \sqrt[n]{c_n} \leq \beta \lim_{n\to\infty} \sup \sqrt[n]{\beta^{-N}} c_N = \beta(1)$

Since $\beta > \alpha$ is arbitrary, we have that $\lim_{n\to\infty} \sup \sqrt[n]{c_n} \leq \alpha$

$\lim_{n\to\infty} \sup \sqrt[n]{c_n} \leq \lim_{n\to\infty} \sup \frac{c_{n+1}}{c_n}.$

2.2 SOME MORE EXAMPLES WITH SOLUTIONS

2.2.1 Example: Investigate the behaviour (convergence or divergence) of $\sum a_n$ if

a) $a_n = \sqrt{n+1} - \sqrt{n};$

b) $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n};$

c) $a_n = \left(\sqrt[n]{n} - 1\right)^n;$

d) $a_n = \frac{1}{1+z^n}$ for complex values of z.

Solution:

(a) Multiplying and dividing a_n by $\sqrt{n+1} + \sqrt{n}$,

we find that $a_n = \frac{1}{\sqrt{n+1}+\sqrt{n}}$, which is larger than $\frac{1}{2\sqrt{n+1}}$.

The series $\sum a_n$ therefore diverges by comparison with the p series $\left(p = \frac{1}{2}\right)$.

Alternatively, since the sum telescopes, the nth partial sum is $\sqrt{n+1} - 1$,

Which obviously tends to infinity.

(b) Using the same trick as in part (a),

We find that $a_n = \frac{1}{n\left[\sqrt{n+1}+\sqrt{n}\right]}$, which is less than $\frac{1}{n^{3/2}}$.

Hence the series converges by comparison with the p series $\left(p - \frac{3}{2}\right)$.

(c) Using the root test, we find that $a_n^{\frac{1}{n}} = \sqrt[n]{n} - 1$, which tends to zero as $n \to \infty$.

Hence the series converges.

(Alternatively, since by part (c) of Known Theorem $\sqrt[n]{n}$ tends to 1 as $n \to \infty$, we have $a_n \leq 2^{-n}$ for all large n, and the series converges by comparison with a geometric series.)

(d) If $|z| < 1$, then $|a_n| \geq \frac{1}{2}$, so that a_n does not tend to zero.

Hence the series diverges.

If $|z| > 1$, the series converges by comparison with a geometric series with $r = \frac{1}{|z|} < 1$.

2.2.2 Example: Prove that the convergence of $\sum a_n$ implies the convergence of $\sum \frac{\sqrt{a_n}}{n}$, if $a_n \geq 0$.

Solution: Since $\left(\sqrt{a_n} - \frac{1}{n}\right)^2 \geq 0$,

It follows that $\frac{\sqrt{a_n}}{n} \leq \frac{1}{2}\left({a_n}^2 + \frac{1}{n^2}\right)$.

Now $\sum {a_n}^2$ converges by comparison with $\sum a_n$ (since $\sum a_n$ converges, we have $a_n < 1$ for large n, and hence ${a_n}^2 < a_n$.

Since $\sum \frac{1}{n^2}$ also converges (p series. $p = 2$),

It follows that $\sum \frac{\sqrt{a_n}}{n}$ converges.

2.2.3 Example: If $\sum a_n$ converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converges.

Solution: We shall show that the partial sums of the series form a Cauchy sequence,

i.e., given $\epsilon > 0$ there exists N such that $|\sum_{k=m+1}^{n} a_k b_k| < \epsilon$ if $n > m \geq N$.

To do this, let $S_n = \sum_{k=1}^{n} a_k$ $(S_0 = 0)$,

so that $a_k = S_k - S_{k-1}$ for $k = 1, 2, \ldots$.

Let M be an upper bound for both $|b_n|$ and $|S_n|$

And let $S = \sum a_n$ and $b = \lim b_n$.

Choose N so large that the following three inequalities hold for all $m > N$ and $n > N$;

$|b_n S_n - bS| < \frac{\epsilon}{3}$; $|b_m S_m - bS| < \frac{\epsilon}{3}$; $|b_m - b_n| < \frac{\epsilon}{3M}$.

Then if $n > m > N$, we have, from the formula for summation by parts,

$\sum_{k=m+1}^{n} a_n b_n = b_n S_n - b_m S_m + \sum_{k=m}^{n-1} (b_k - b_{k+1}) S_k$

Our assumptions yield immediately that $|b_n S_n - b_m S_m| < \frac{2\epsilon}{3}$, and

$|\sum_{k=m}^{n-1} (b_k - b_{k+1}) S_k| \leq M \sum_{k=m}^{n-1} |b_k - b_{k+1}|$.

Since the sequence $\{b_n\}$ is monotonic, we have

$$\sum_{k=m}^{n-1} |b_k - b_{k+1}| = |\sum_{k=m}^{n-1} (b_k - b_{k+1})| = |b_m - b_n| < \frac{\epsilon}{3M}.$$

From which the desired inequality follows.

2.3 SUMMARY:

This lesson focuses on helping learners comprehend numerical series, including convergence and divergence, and apply series properties to solve mathematical problems. Highlights of this lesson are Definitions and theorems of numerical series, Convergence and divergence of series, Series properties and applications and Examples with solutions and exercises.

2.4 TECHNICAL TERMS:

- Binomial
- Bounded above
- Convergence
- Divergence
- Geometric Series
- Infinite Series
- Irrational
- Monotonic

2.5 SELF ASSESSMENT QUESTIONS:

1. Investigate the behaviour (convergence or divergence) of $\sum a_n$ if

a) $a_n = \sqrt{n+1} - \sqrt{n}$

b) $a_n = (\sqrt[n]{n} - 1)^n$

2. If $\sum a_n$ converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converges.

3. $\sum \left[\sqrt{(n^3+1)} - \sqrt{n^3}\right]$.

4. $\sum \left[\sqrt{(n^2+1) - n}\right]$.

2.6 SUGGESTED READINGS:

1. Principles of Mathematical Analysis, Third Edition, Mc Graw-Hill International Editions Walter Rudin.

2. Mathematical Analysis by Tom M. Apostol, Narosa Publishing House, 2nd Edition, 1985.

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LESSON-3

POWER SERIES AND MULTIPLICATION OF SERIES

OBJECTIVES:

The objective of the lesson is to help the learners to understand

- To understand the concept of Power series and Multiplication of series including convergence and divergences.
- To apply Power series and Multiplication of series properties for solving mathematical Problems.

STRUCTURS:

3.1 POWER SERIES
3.2 MULTIPLICATION OF SERIES
3.3 SOME MORE EXAMPLES WITH SOLUTIONS
3.4 SUMMARY
3.5 TECHNICAL TERMS
3.6 SELF ASSESSMENT QUESTIONS
3.7 SUGGESTED READINGS

3.1 POWER SERIES:

3.1.1 Theorem: Given 'power series' $\sum a_n z^n (a_n \in \varphi)$

Let $\alpha = \lim \sup |a_n|^{\frac{1}{n}}$ and $R = \frac{1}{\alpha}$

Here $R = \{0 \text{ if } \alpha = \infty \text{ and } \infty \text{ if } \alpha = 0\}$

Then $\sum a_n z^n$ converges for $|z| < R$, and diverges for $|z| > R$.

Proof: Applying the root test to $a_n z^n$

$$\lim |a_n z^n|^{\frac{1}{n}} = |z| \lim |a_n|^{\frac{1}{n}}$$
$$= |z|/R$$

3.1.2 Note: We call R, the radius of convergence of $\sum a_n z^n$. For $|z| = R$, we can't say anything define.

3.1.3 Example: Consider the series $\sum_{n=1}^{\infty} n^n z^n$. Find the radius of convergence of the series.

Solution: Here $c_n = n^n \ \forall n \geq 1$

Now $\alpha = \lim_{n\to\infty} sup \sqrt[n]{|c_n|} = \lim_{n\to\infty} sup \sqrt[n]{|n^n|} = \lim_{n\to\infty} \sup n = \infty$

$\Rightarrow \alpha = \infty$

Therefore the radius of convergence of the series $R = \frac{1}{\alpha} = \frac{1}{\infty} = 0$.

3.1.4 Example: Consider the series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$.

Solution: Here $c_n = \frac{1}{n!} \forall n \geq 1, c_{n+1} = \frac{1}{(n+1)!}$

Now $\left|\frac{c_{n+1}}{c_n}\right| = \left|\frac{\left(\frac{1}{(n+1)!}\right)}{\left(\frac{1}{n!}\right)}\right| = \left|\frac{1}{(n+1)}\right|$

$\alpha = \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty} \left|\frac{1}{(n+1)}\right| = 0$

$\Rightarrow \alpha = 0$

Therefore the radius of convergence of the series $R - \frac{1}{\alpha} - +\infty$.

3.1.5 Example: Consider the series $\sum_{n=1}^{\infty} \frac{z^n}{n}$.

Solution: Here $c_n = \frac{1}{n} \forall n > 1$

Now $\alpha = \lim_{n\to\infty} \sqrt[n]{|c_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{n}\right|} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1$

$\Rightarrow \alpha = 1$

Therefore the radius of convergence of the series $R = \frac{1}{\alpha} = \frac{1}{1} = 1$.

3.1.6 Theorem: [Abel's Partial summation formula]

Given sequence $\{a_n\}$ and $\{b_n\}$, Let $A_n = \sum a_k (A_{-1} = 0)$. Then for $0 \leq p \leq q$,

$\sum a_n b_n = \sum A_n (b_n \quad b_{n+1}) \quad A_{p-1} b_p \ | \ A_q b_q$

Proof: $\sum a_n b_n = \sum (A_n - A_{n-1}) b_n$

$= \sum A_n b_n - \sum A_{n-1} b_n$

$= \sum A_n b_n - \sum A_n b_{n+1}$

$= \sum A_n (b_n - b_{n+1}) - \sum A_{p-1} b_p + A_q b_q$.

3.1.7 Theorem: (Dirichlet).

Let (1) the partial sums A_n of $\sum a_n$ be bounded.

(2) $b_n \downarrow 0$

Then $\sum a_n b_n$ converges.

Proof: Let $|A_n| \leq M [] \ n$. Given $\epsilon > 0$, choose $b_N \ni b_N \leq \epsilon / 2M$. For $N \leq p \leq q$,

$\sum A_n b_n = \left|\sum (A_n (b_n - b_{n+1}) - A_{p-1} b_p + A_q b_q)\right|$

$\leq M \left|\sum \left((b_n - b_{n+1}) + b_p + b_q\right)\right|$

$\leq 2M b_p$

$\leq 2M b_N$

$< \epsilon$

[Notice $b_n - b_{n+1} \geq 0$]

By Cauchy Criterion, $\sum a_n b_n$ converges.

3.1.8 Corollary: (Leibnitz Test)

If $C_n \downarrow 0, \sum (-1)^{n-1} C_n$ converges.

Proof: Take $a_n = (-1)^n$ in Dirichlet's test

Then $A_n = \{0$ if n is even so $|A_n| \leq 1$ and -1 if n is odd$\}$

3.1.9 Theorem: Suppose

(i) $|c_1| \geq |c_2| \geq \cdots$

(ii) $c_{2m-1} \geq 0, c_{2m} \leq 0 \ (m = 1,2 \ldots.)$

(iii) $\lim_{n\to\infty} c_n - 0$. Then $\sum_{n=1}^{\infty} c_n$ converges.

Proof: Suppose

(i) $|c_1| \geq |c_2| \geq \cdots$

(ii) $c_{2m-1} \geq 0, c_{2m} \leq 0 \ (m = 1,2 \ldots.)$ and

(iii) $\lim_{n\to\infty} c_n = 0$.

For integer $n \geq 1$, write $a_n = (-1)^{n+1}$ and $b_n = |c_n|$

Now the partial sums sequence of series $\sum_{n=1}^{\infty} a_n$. form a bounded sequence and

$b_1 \geq b_2 \geq b_3 \geq \cdots$ and $\lim_{n\to\infty} b_n = 0$

So the above theorem $\sum_{n=1}^{\infty} a_n b_n$ converges.

$\Rightarrow \sum_{n=1}^{\infty} c_n$ converges.

3.1.10 Definition: The series which satisfies the condition (ii) in the above theorem is called as alternating series.

3.1.11 Theorem: Suppose the radius of convergence of $\sum_{n=0}^{\infty} c_n z^n$ is 1 and suppose $c_0 \geq c_1 \geq c_2 \geq \cdots \lim_{n\to\infty} c_n = 0$. Then $\sum_{n=0}^{\infty} c_n z^n$ converges at every point on the circle $|z| = 1$, except possibly at $z = 1$.

Proof: Suppose that the series $\sum_{n=0}^{\infty} c_n z^n$ converges for all z such that $|z| < 1$.

Also suppose that $c_0 \geq c_1 \geq c_2 \geq \cdots \lim_{n\to\infty} c_n = 0$

For any integer $n \geq 0$, write $a_n = z^n$ and $b_n = c_n$

For $n \geq 0$, let $A_n = \sum_{k=0}^{n} a_k$

Then $|A_n| = |a_0 + a_1 + \cdots + a_n| = |1 + z + z^2 + \cdots + z^n|$

$= \left|\frac{1-z^{n+1}}{1-z}\right| \leq \frac{2}{|1-z|}$ if $|z| = 1$ and $z \neq 1$

Therefore $\forall n \geq 0, |A_n| \leq M$ where $M = \frac{2}{|1-z|}$ if $|z| = 1$ and $z \neq 1$

So the partial sums A_n of $\sum_{k=0}^{n} a_k$ form a bounded sequence and

$b_1 \geq b_2 \geq b_3 \geq \cdots$ and $\lim_{n\to\infty} b_n = 0$

So the known theorem $\sum_{n=0}^{\infty} a_n b_n$ converges.

Here, $\sum_{n=0}^{\infty} c_n z^n$ converges if $|z| = 1$ and $z \neq 1$.

3.1.12 Theorem: If $\sum |a_n|$ converges, then $\sum a_n$ converges.

Proof: Given $\epsilon > 0, \exists N \ni$ for $n \geq m \geq N$

$|\sum A_n| < \epsilon$.

Since $|\sum a_k| < \epsilon$, $\sum a_n$ converges.

3.1.13 Remarks: If $\sum |a_n|$ converges, we say $\sum a_n$ converges absolutely, for $\sum a_n$ $(a_n \geq 0)$, absolute converges is same as convergence.

If $\sum a_n$ converges, $\sum |a_n|$ need not converges $\sum (-1)^{\frac{1-n}{n}}$ converges by Leibnitz test, but not absolutely.

3.1.14 Theorem: If $\sum a_n = A$ and $\sum b_n = B$, then $\sum(a_n + b_n) = A + B$ and $\sum ca_n = cA$ for any fixed $'c'$.

Proof: Suppose that $\sum a_n = A$ and $\sum b_n = B$

For $n \geq 0$, write $A_n = \sum_{k=0}^{n} a_k$ and $B_n = \sum_{k=0}^{n} b_k$

Then $\{A_n\}$ and $\{B_n\}$ are the sequences of partial sums of $\sum a_n$ and $\sum b_n$ respectively.

Also for $n \geq 0$, $A_n + B_n = \sum_{k=0}^{n} a_k + \sum_{k=0}^{n} b_k = \sum_{k=0}^{n} (a_k + b_k)$

So, $\{A_n + B_n\}$ is a sequence of partial sums of the series $\sum_{n=0}^{\infty}(a_n + b_n)$

Since $\sum a_n = A$ and $\sum b_n = B$

we have that $\lim_{n\to\infty} A_n = A$ and $\lim_{n\to\infty} B_n = B$

So by a known theorem, $\lim_{n\to\infty}(A_n + B_n) = A + B$ and

$\lim_{n\to\infty}(cA_n) = c \lim_{n\to\infty} A_n = cA$, where c is fixed constant.

Hence, $\sum(a_n + b_n) = A + B$ and $\sum ca_n = cA$ for any fixed $'c$

3.2 MULTIPLICATION OF SERIES:

3.2.1 Definition: Given $\sum_{k=0}^{n} a_n$ and $\sum b_n$, we define $C_n = \sum a_k b_{n-k}$ and we call $\sum C_n$, the Cauchy product of $\sum a_n$ and $\sum b_n$.

3.2.2 Remark: If $\sum a_n$ and $\sum b_n$ converges, the Cauchy product of $\sum a_n$ and $\sum b_n$ need not converge.

Let $a_n = b_n = \frac{(-1)n}{\sqrt{(n+1)}}$, $\sum a_n$ and $\sum b_n$ converges by Leibnitz test.

$$C_n = \sum a_k b_{n-k} = \sum \left[\frac{(-1)^k}{\sqrt{(n+1)}} \cdot \frac{(-1)^{n-k}}{\sqrt{(n-k+1)}}\right]$$

$$\Rightarrow |C_n| > \sum \left[\frac{1}{\sqrt{(n+1)}} \cdot \frac{1}{\sqrt{(n-k+1)}}\right] = 1$$

So that $C_n \to 0$.

Hence $\sum C_n$ doesn't converges.

3.2.3 Theorem: (Mertens)

If $\sum a_n$ converges absolutely to A and $\sum b_n$ converges and $\sum C_n = AB$.

Proof: Let $A_n = \sum a_k$, $B_n = \sum b_k$ and $C_n = \sum c_k$

$\beta_n - B_n - B$.

$C_n = C_0 + C_1 + \cdots \ldots + C_n$

$= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0)$

$= a_0 B_n + a_1 B_{n-1} + \cdots + a_n B_0$

$= a_0 (\beta_n + B) + a_1 (\beta_{n-1} + B) + \cdots + a_n (\beta_0 + B)$

$= A_n B + a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0$

Let $\gamma_n = a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0$

We will show that $\gamma_n \to 0$ so that $C_n = A_n B \to AB$ which proves the theorem.

Let $\alpha = \sum a_n$

Since $\beta_n \to 0$, given $\epsilon > 0$, choose $N \ni |\beta_n| < \epsilon \; \forall \; n > N$.

So $|\gamma_n| \leq |\beta_0 a_n + \beta_1 a_{n-1} + \cdots + \beta_N a_{n-N}| + |\beta_{N+1} a_{N-1} + \cdots + \beta_n C_0|$

$\leq |\beta_0 a_n + \cdots + \beta_N a_{n-N}| + \epsilon\alpha$

Fix N and let $n \to \infty$

Then $\lim |\gamma_n| \leq \epsilon\alpha$ since $a_n \to 0$ [$\sum |a_n|$ converges]

Let $\epsilon \to 0$. Then $\gamma_n \to 0$ and this completes the proof.

3.3 SOME MORE EXAMPLES WITH SOLUTIONS:

3.3.1 Example: Find the radius of convergence of each of the following series

(a) $\sum n^3 z^n$

(b) $\sum \frac{2^n}{n!} z^n$

(c) $\sum \frac{2^n}{n^2} z^n$

(d) $\sum \frac{n^3}{3^n} z^n$

Solution: (a) The radius of convergence is 1,

Since $a_n = n^3$ satisfies $\lim\limits_{n\to\infty} \frac{a_n}{a_{n+1}} = 1$

(b) The radius of convergence is infinite,

Since $a_n = \frac{2^n}{n!}$ satisfies $\lim\limits_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim\limits_{n\to\infty} \frac{n+1}{2} = \infty$

(c) The radius of convergence is $\frac{1}{2}$,

Since $a_n = \frac{2^n}{n^2}$ satisfies $\lim\limits_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim\limits_{n\to\infty} \frac{1}{2}\left(1 + \frac{1}{n}\right)^2 = \frac{1}{2}$.

(d) The radius of convergence is 3,

Since $a_n = \frac{n^3}{3^n}$ satisfies $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} 3\left(\frac{n}{n+1}\right)^3 = 3.$

3.3.2 Example: Suppose that the coefficients of the power series $\sum a_n z^n$ are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most 1.

Solution: The series diverges if $|z| > 1$, since its general term does not tend to zero. (Infinitely many terms are larger than 1 in absolute value).

3.3.3 Example: Suppose $a_n > 0$, $s_n = a_1 + \cdots + a_n$, and $\sum a_n$ diverges.

(a) Prove that $\sum \frac{a_n}{1+a_n}$ diverges.

(b) Prove that $\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$ and deduce that $\sum \frac{a_n}{s_n}$ diverges.

(c) Prove that $\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$ and deduce that $\sum \frac{a_n}{s_n}$ diverges.

(d) What can be said about $\sum \frac{a_n}{1+na_n}$ and $\sum \frac{a_n}{1_n^2 a_n}$?

Solution: (a) If a_n does not remain bounded,

then $\frac{a_n}{1+a_n}$ does not tend to zero,

and hence the series $\sum \frac{a_n}{1+a_n}$ diverges.

If $a_n \leq M$ for all n, then $\frac{a_n}{1+a_n} \geq \frac{1}{1+M} a_n$,

And hence again the series is divergent.

(b) Replacing each denominator on the left by S_{N+k},

We have $\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq \frac{1}{s_{N+k}}(a_{N+1} + a_{N+2} + \cdots + a_{N+k})$

$$-\frac{1}{S_{N+k}}(S_{N+k} - S_N)$$

$$= 1 - \frac{S_N}{S_{N+k}}$$

It follows that the partial sums of the series $\sum \frac{a_n}{S_n}$ do not form a Cauchy sequence. For, no matter how large N is taken, if N is held fixed, the right hand side can be made larger than $\frac{1}{2}$ by taking k sufficiently large (since $S_{N+k} \to \infty$).

(c) We observe that if $n \geq 2$, then

$$\frac{1}{S_{n-1}} - \frac{1}{S_n} = \frac{S_n - S_{n-1}}{S_{n-1}S_n} = \frac{a_n}{S_{n-1}S_n} \geq \frac{a_n}{s_n^2}$$

Since the series $\sum_{n=2}^{\infty} \frac{1}{S_{n-1}} - \frac{1}{S_n}$ converges to $\frac{1}{a_1}$,

It follows by comparison that $\sum \frac{a_n}{s_n^2}$ converges.

(d) The series $\sum \frac{a_n}{1+na_n}$ may be either convergent or divergent.

If the sequence $\{na_n\}$ is bounded above or has a positive lower bound, it definitely diverges.

Thus if $na_n \leq M$, each term is at least $\frac{1}{1+M} a_n$, and so the series diverges.

If $na_n \geq \epsilon > 0$ for all n, then each term is at least $\frac{\epsilon}{1+\epsilon} \frac{1}{n}$, and once again the series is divergent.

In general, however, the series $\sum \frac{a_n}{1+na_n}$ may converge.

For example let $a_n = \frac{1}{n^2}$ if n is not a perfect square and

$a_n = \frac{1}{\sqrt{n}}$ if n is a perfect square.

The sum of $\frac{a_n}{1+na_n}$ over the non squares obviously converges by comparison with the p series, $p = 2$.

As for the sum over the square integers it is $\sum \frac{1}{n+n^2}$, which converges by comparison with the p series, $p = 2$.

Finally, the series $\sum \frac{a_n}{1+n^2 a_n}$ is obviously majorized by the p series, $p = 2$, hence converges.

3.3.4 Example: Suppose $a_n > 0$ and $\sum a_n$ converges. Put $r_n = \sum_{m=n}^{\infty} a_n$.

(a) Prove that $\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$ if $m < n$, and deduce that $\sum \frac{a_n}{r_n}$ diverges.

(b) Prove that $\frac{a_n}{\sqrt{r_n}} < 2\left(\sqrt{r_n} - \sqrt{r_{n+1}}\right)$ and deduce that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

Solution: (a) Replacing all the denominators on the left-hand side by the largest one (r_m),

We find $\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > \frac{a_m + \cdots + a_n}{r_m} = \frac{r_m - r_{n+1}}{r_m} > 1 - \frac{r_n}{r_m}$

Since $r_n > r_{n+1}$.

As in the previous problem, this keeps the partial sums of the series $\sum \frac{a_n}{r_n}$ from forming a Cauchy sequence.

No matter how large m is taken, one can choose n larger so that the difference $\sum_{k=m}^{n} \frac{a_k}{r_k}$ is at least $\frac{1}{2}$, since $r_n \to 0$ as $n \to \infty$.

(b) We have $\frac{a_n}{\sqrt{r_n}}\left(\sqrt{r_n} + \sqrt{r_{n+1}}\right) = a_n + a_n \frac{\sqrt{r_{n+1}}}{\sqrt{r_n}} < 2a_n = 2(r_n - r_{n+1})$.

Dividing both sides by $\sqrt{r_n} + \sqrt{r_{n+1}}$ now yields the desired inequality.

Since the series $\sum\left(\sqrt{r_n} - \sqrt{r_{n+1}}\right)$ converges to $\sqrt{r_1}$,

It follows by comparison that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

3.3.5 Example: Prove that the Cauchy product of two absolutely convergent series converges absolutely.

Solution: Since both the hypothesis and conclusion refer to absolute convergence, we may assume both series consist of non negative terms.

We let $S_n = \sum_{k=0}^{n} a_n$, $T_n = \sum_{k=0}^{n} b_n$ and $U_n = \sum_{k=0}^{n} \sum_{l=0}^{k} a_l\, b_{k-l}$.

We need to show that U_n remains bounded, given that S_n and T_n are bounded.

To do this we make the convention that $a_{-1} = T_{-1} = 0$, in order to save ourselves from having to separate off the first and last terms when we sum by parts.

We then have $U_n = \sum_{k=0}^{n} \sum_{l=0}^{k} a_l\, b_{k-l}$

$$= \sum_{k=0}^{n} \sum_{l=0}^{k} a_l \left(T_{k-l} - T_{k-l-1}\right)$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{k} a_{k-j} \left(T_j - T_{j-1}\right)$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{k} (a_{k-j} - a_{k-j-1}) T_j$$

$$= \sum_{j=0}^{n} \sum_{k=j}^{n} (a_{k-j} - a_{k-j-1}) T_j$$

$= \sum_{j=0}^{n} a_{n-j} T_j$

$\leq T \sum_{m=0}^{n} a_m$

$= TS_n$

$\leq ST$

Thus U_n is bounded, and hence approaches a finite limit.

3.3.6 Example: If $\{s_n\}$ is a complex sequence, define its arithmetic mean $\sigma_n = 0$ by $\sigma_n = \frac{s_0+s_1+\cdots+s_n}{n+1}$ $(n = 0,1,2, \dots)$.

(a) If $\lim s_n = s$, prove that $\lim \sigma_n = s$.

(b) Construct a sequence $\{s_n\}$ which does not converge, although $\lim \sigma_n = 0$.

(c) Can it happen that $s_n > 0$ for all n and that $\limsup s_n = \infty$, even though $\lim \sigma_n = 0$?

(d) Put $a_n = s_n - s_{n-1}$ for $n \geq 1$. Show that $s_n - \sigma_n = \frac{1}{n+1}\sum_{k=1}^{n} ka_k$.

Assume that $\lim(na_n) = 0$ and that $\{\sigma_n\}$ converges. Prove that $\{s_n\}$ converges. [This gives a converse of (a), but under the additional assumption that $na_n \to 0$.]

(e) Derive the last conclusion from a weaker hypothesis: Assume $M < \infty$, $|na_n| \leq M$ for all n, and $\lim \sigma_n = \sigma$. Prove that $\lim s_n = \sigma$ by completing the following outline:

If $m < n$, then

$$s_n - \sigma_n = \frac{m+1}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m}\sum_{i=m+1}^{n}(s_n - s_i).$$

For these i,

$$|s_n - s_i| \leq \frac{(n-i)M}{i+1} \leq \frac{(n-m-1)M}{m+2}.$$

Fix $\epsilon > 0$ and associate with each n the integer m that satisfies

$$m \leq \frac{n-\epsilon}{1+\epsilon} < m+1.$$

Then $(m+1)/(n-m) \leq 1/\epsilon$ and $|s_n - s_i| < M\epsilon$.

Hence $\lim_{n\to\infty} \sup|s_n - \sigma| \leq M\epsilon$.

Since ϵ was arbitrary, $\lim s_n = \sigma$.

Solution: Let $\epsilon > 0$.

Let $M = \sup\{|s_n|\}$, and

let N_0 be the first integer such that $|s_n - s| < \frac{\epsilon}{2}$ for all $n > N_0$.

Let $N = \max\left(N_0, \left[\frac{2(N_0+1)(M+|s|)}{\epsilon}\right]\right)$,

Then if $n > N$, we have

$$|\sigma_n - s| = \left|\frac{(s_0 - s) + (s_1 - s) + \cdots + (s_n - s)}{n+1}\right|$$

$$\leq \left|\frac{(s_0 - s) + \cdots + (s_{N_0} - s)}{n+1}\right| + \left|\frac{(s_{N_0+1} - s) + (s_n - s)}{n+1}\right|$$

The first sum on the right-hand side here is at most $\frac{(N_0+1)(M+|s|)}{n+1}$, and since $n + 1 > \frac{2(N_0+1)(M+|s|)}{\epsilon}$, this sum is at most $\frac{\epsilon}{2}$.

The second sum is at most $\frac{(n \quad N_0)\frac{\epsilon}{2}}{n+1}$, which is at most $\frac{\epsilon}{2}$.

Thus $|\sigma_n - s| < \epsilon$ if $n > N$, which was to be proved.

(b) Let $s_n = (-1)^n$.

Here σ_n is 0 if n is odd and $\frac{1}{n+1}$ if n is even.

Thus $\sigma_n \to 0$, though s_n has no limit.

(c) Let $s_n = \frac{1}{n}$ if n is not a perfect cube and $s_n = \sqrt[3]{n}$ if n is a perfect cube.

Then if $k^3 \leq n < (k+1)^3$ we have

$$\sigma_n \leq \frac{1}{n+1}\sum_{m=1}^{n} \frac{1}{m} + \frac{1}{n+1}\sum_{j=1}^{k} j$$

$$= \frac{1}{n+1}\left(\sum_{m=1}^{n} \frac{1}{m}\right) + \frac{1}{n+1}\frac{k(k+1)}{2}$$

The first sum on the right tends to zero by part (a) applied to the sequence $s_0 = 0$, $s_n = \frac{1}{n}$ for $n \geq 1$.

As the last term, since $n > k^3$, it is less than $\frac{1}{2k} + \frac{1}{2k^2}$, which tends to zero as $k \to \infty$.

Since $(k+1)^3 > n$, it follows that k tends to infinity as n tends to infinity, and hence we have $\sigma_n \to 0$, even though $s_{n^3} \to \infty$.

(d) If we set $a_0 = s_0$, we have $s_n = \sum_{k=0}^{n} a_k$.

Then $s_n - \sigma_n = s_n - \frac{s_0+s_1+\cdots+s_n}{n+1}$

$$= (a_0 + a_1 + \cdots + a_{n-1} + a_n) - \frac{(n+1)a_0 + na_1 + \cdots + 2a_{n-1} + a_n}{n+1}$$

$$= \frac{a_1 + 2a_2 + \cdots + (n-1)a_{n-1} + na_n}{n+1}$$

Which was to be proved.

If $na_n \to 0$, then the expression on the right-hand side tends to zero by part (a) with s_n replaced by na_n. Hence $s_n - \sigma_n \to 0$.

(e) If $m < n$ we have

$$\sigma_n - \sigma_m = \frac{s_0 + \cdots + s_n}{n+1} + \frac{s_0 + \cdots + s_m}{m+1}$$

$$= (s_0 + \cdots + s_n)\left(\frac{1}{n+1} - \frac{1}{m+1}\right) + \sum_{i=m+1}^{n} \frac{s_i}{m+1}$$

$$= \frac{m \quad n}{m+1}\sigma_n + \frac{1}{m+1}\sum_{i=m+1}^{n} s_i$$

If we multiply both sides of this equation by $\frac{m+1}{m-n}$, and then transpose the left-hand side to the right and the term υ_n to the left, we obtain

$$-\sigma_n = \frac{m+1}{m-n}(\sigma_n - \sigma_m) - \frac{1}{n-m}\sum_{i=m+1}^{n} s_i$$

Adding $s_n = \frac{1}{n-m}\sum_{i=m+1}^{n} s_n$ to both sides then yields the result.

We then have

$|s_n - s_i| = |a_{i+1} + \cdots + a_n| \le M\left(\frac{1}{i+1} + \cdots + \frac{1}{n}\right) \le \frac{(n-i)M}{i+1}$.

Since the function $\frac{n-x}{x+1} = \frac{n+1}{x+1} - 1$ is decreasing, the maximal value of the right hand side here is reached with $i = m+1$, so that $|s_n - s_i| \le \frac{(n-m-1)M}{m+2}$ as asserted.

When we choose m to be the largest integer in $\frac{n-\epsilon}{1+\epsilon}$, we clearly have $m < n$.

Since ϵ is fixed, we can assume $m > \epsilon$.

The inequality $\frac{n-\epsilon}{1+\epsilon} < m+1$ can easily be converted to $\frac{(n-m-1)M}{m+2} < \epsilon$, and the inequality $m \leq \frac{n-\epsilon}{1+\epsilon}$ likewise becomes $\frac{m+1}{n-m} \leq \frac{1}{\epsilon}$.

The first of these implies that $m \to \infty$ as $n \to \infty$, and we have

$|s_n - \sigma_n| \leq \frac{1}{\epsilon}|\sigma_n - \sigma_m| + M\epsilon$ for all n.

This implies that the limit of any subsequence of $|s_n - \sigma_n|$ is at most $M\epsilon$, and since ϵ is arbitrary, every convergent subsequence of $|s_n - \sigma_n|$ converges to zero.

This, of course, implies that $s_n - \sigma_n$ tends to zero, so that if $\sigma_n \to s$, then $s_n \to s$.

Exercise For Lesson-2 & Lesson-3

Test for convergence, the following series:

1. (i) $\sum\left(1+\frac{1}{n}\right)^{n^2}$ (ii) $\sum\left(1+\frac{1}{n}\right)^{-n^2}$

Ans: (i) Divergent (ii) Convergent

2. $\sum\left[\sqrt{(n^2+1)}-n\right]$

Ans: Divergent

3. $\sum\left[\sqrt{(n^3+1)}-\sqrt{n^3}\right]$

Ans: Convergent

4. $\frac{1.2}{3^2.4^2}+\frac{3.4}{5^2.6^2}+\frac{5.6}{7^2.8^2}+\cdots$

Ans: Convergent

5. $\frac{1}{1+2^{-1}}+\frac{2}{1+2^{-2}}+\frac{3}{1+2^{-3}}+\cdots$

Ans: Divergent

6. $\sum\frac{n^p}{(n+1)^p}$

Ans: Convergent if $p-q+1>0$ and Divergent if $p-q+1\leq 0$

7. $\sum\frac{n^3}{(n-1)!}$

Ans: Convergent

8. $\sum\left(\frac{n^3+a}{2^n+a}\right)$

Ans: Convergent

9. (i) $\sum\cos\frac{1}{n}$ (ii) $\sum\sin\frac{1}{n}$

Ans: Both Divergent

10. $1+\frac{x}{2^2}+\frac{x^2}{3^2}+\frac{x^3}{4^2}+\cdots$

Ans: Convergent if $x \le 1$, divergent if $x > 1$.

11. $\sum\left(\frac{x^n}{x+n}\right)$

Ans: Convergent if $x < 1$, divergent if $x \ge 1$

12. $1+\frac{2^p}{2!}+\frac{3^p}{3!}+\frac{4^p}{4!}+\cdots$

Ans: Convergent

13. $\frac{2}{1^2}x+\frac{3^2}{2^3}x^2+\frac{4^3}{3^4}x^3+\cdots+\frac{(n+1)^n}{n^{n+1}}x^n+\cdots$

Ans: Convergent if $x < 1$, and divergent if $x \ge 1$.

14. Show that the series $1+\frac{\alpha+1}{\beta+1}+\frac{(\alpha+1)(2\alpha+1)}{(\beta+1)(2\beta+1)}+\frac{(\alpha+1)(2\alpha+1)(3\alpha+1)}{(\beta+1)(2\beta+1)(3\beta+1)}+\cdots$ converges if $\beta>\alpha>0$ and diverges if $\alpha \ge \beta > 0$.

15. Test for convergence the series whose n^{th} terms are (i) $\frac{1}{x^n+x^{-n}}$ (ii) $\frac{a^n}{x^n+a^n}$

Ans: (i) Convergent if $x > 1$ or $x < 1$, divergent if $x = 1$.

(ii) Convergent if $x > a$ and divergent if $x \le a$.

16. $\frac{x}{1}+\frac{1}{2}\frac{x^3}{3}+\frac{1}{2}\frac{3}{4}\frac{x^5}{5}+\frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{x^7}{7}+\cdots$

Ans: Convergent if $x^2 \le 1$, Divergent if $x^2 > 1$

17. $1+\frac{2^2}{3^2}+\frac{2^2}{3^2}\frac{4^2}{5^2}+\frac{2^2}{3^2}\frac{4^2}{5^2}\frac{6^2}{7^2}+\cdots$

Ans: Divergent

18. $\frac{1^2}{2^2}+\frac{1^2}{2^2}\frac{3^2}{4^2}x+\frac{1^2}{2^2}\frac{3^2}{4^2}\frac{5^2}{6^2}x^2+\cdots$

Ans: Convergent if $x < 1$ and divergent if $x \ge 1$.

19. $x+\frac{2^2x^2}{2!}+\frac{3^3x^3}{3!}+\frac{4^4x^4}{4!}+\cdots$

Ans: Convergent if $x < \frac{1}{e}$, divergent if $x \ge \frac{1}{e}$

20. $x+\frac{1}{2}\frac{x^2}{4}+\frac{1.3.5}{2.4.6}\frac{x^4}{8}+\frac{1.3.5.7.9}{2.4.6.8.10}\frac{x^6}{12}+\cdots$

Ans: Convergent if $x^2 \le 1$ and divergent if $x^2 > 1$

21. $1+a+\frac{a(a+1)}{1.2}+\frac{a(a+1)(a+2)}{1.2.3}+\cdots$

Ans: Convergent if $a \le 0$ and divergent if $a > 0$

22. $\frac{a}{a+3}+\frac{a(a+2)}{(a+3)(a+5)}x+\frac{a(a+2)(a+4)}{(a+3)(a+5)(a+7)}x^2+\cdots$

Ans: Convergent if $x \le 1$ and divergent if $x > 1$

23. $\frac{1^2}{4^2}+\frac{1^2}{4^2}\frac{5^2}{8^2}x+\frac{1^2}{4^2}\frac{5^2}{8^2}\frac{9^2}{12^2}x^2+\cdots$

Ans: Convergent

24. $1+\frac{a}{1.\beta}x+\frac{a(a+1)^2}{1.2\beta(\beta+1)}x^2+\frac{a(a+1)^2(a+2)^2}{1.2.3\beta(\beta+1)(\beta+2)}x^3+\cdots$

Ans: Convergent if $x < 1$ and divergent if $x > 1$ and when $x = 1$, then convergent if $\beta > 2\alpha$ and divergent if $\beta \le 2\alpha$

25. $\frac{a+x}{1!}+\frac{(a+2x)^2}{2!}+\frac{(a+3x)^2}{3!}+\cdots$

Ans: Convergent if $x<\frac{1}{e}$, divergent if $x\geq\frac{1}{e}$

26. If $\frac{u_n}{u_{n+1}}=\frac{n^k+An^{k-1}+Bn^{k-2}+Cn^{k-3}}{n^k+an^{k-1}+bn^{k-2}+cn^{k-3}}$ where k is a positive integer, show that the series $\sum u_n$ is convergent if $A-a-1$ is positive and divergent if $A-a-1$ is negative or zero.

3.4 SUMMARY:

This lesson provides a comprehensive introduction to Power series and Multiplication of series including Convergence and divergence. Key concepts are defined, and theorems are supported with proofs. Additionally, examples are provided to illustrate the applications of these concepts.

3.5 TECHNICAL TERMS:

- Alternating series
- Bounded sequence
- Converges
- Diverges
- Limit
- Partial summation
- Power Series
- Radius
- Supremum

3.6 SELF ASSESSMENT QUESTIONS:

1. Find the radius of convergence of each of the following series

a) $\sum n^3 z^n$

b) $\sum \frac{2^n}{n!} z^n$

2. Suppose that the coefficients of the power series $\sum a_n z^n$ are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most 1.

3. Prove that $\frac{a_{N+1}}{S_{N+1}}+\cdots+\frac{a_{N+k}}{S_{N+k}}\geq 1-\frac{S_N}{S_{N+k}}$ and deduce that $\sum\frac{a_n}{S_n}$ diverges.

4. What can be said about $\sum\frac{a_n}{1+na_n}$ and $\sum\frac{a_n}{1_n^2 a_n}$?

5. $x+\frac{1}{2}\frac{x^2}{4}+\frac{1.3.5}{2.4.6}\frac{x^4}{8}+\frac{1.3.5.7.9}{2.4.6.8.10}\frac{x^6}{12}+\cdots$ Convergent if $x^2\leq 1$ and divergent if $x^2>1$. Is it true or false.

6. $\frac{a+x}{1!}+\frac{(a+2x)^2}{2!}+\frac{(a+3x)^2}{3!}+\cdots$ Convergent if $x<\frac{1}{e}$, and divergent if $x\geq\frac{1}{e}$. Is it true or false.

3.7 SUGGESTED READINGS:

1. Principles of Mathematical Analysis, Third Edition, Mc Graw-Hill International Editions Walter Rudin.

2. Mathematical Analysis by Tom M. Apostol, Narosa Publishing House, 2nd Edition, 1985.

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LESSON-4

LIMITS OF FUNCTIONS AND CONTINUOUS FUNCTIONS ON METRIC SPACES

OBJECTIVES:

The objective of the lesson is to help the learners to understand

- To understand the concepts of limit and continuity of functions on metric spaces that is essential in the study of real world applications such as modeling motion, force and energy.
- To develop problem solving skills in calculus and analysis.

STRUCTURE:

4.1 INTRODUCTION
4.2 LIMITS OF FUNCTIONS
4.3 CONTINUOUS FUNCTIONS
4.4 SUMMARY
4.5 TECHNICAL TERMS
4.6 SELF ASSESSMENT QUESTIONS
4.7 SUGGESTED READINGS

4.1 INTRODUCTION:

In this lesson the notion of metric space, open set, closed set, compact set connected and limit of a function from one metric space into another is introduced. If X and Y are metric spaces and $E \subseteq X$ and f maps E into y and p is a limit point E, then $\lim_{x\to\infty} f(x) = q$ if and only if $\lim_{n\to\infty} f(p_n) = q$ for every sequence $\{p_n\}$ in E such that $P_n \neq P$ for all n and $\lim_{n\to\infty} f(p_n) = p$ is proved. Next the continuity of a function from a metric space into a metric space is defined. It has also been proved that if X and Y are metric spaces, $E \subseteq X$ and f maps E into Y and if $p \in E$ is a limit point of E, then f is continuous at P if and only if $\lim_{x\to p} f(x) = f(p)$. Further it is proved that a mapping f of a metric space X into a metric space Y is continuous on X if and only if $f^{-1}(V)$ is open in X for every open set V in Y.

4.1.1 Definition: Let J denote the set of all positive integers and for any n, J_n be the set of integers $1,2,3,,n$

A Set E is said to be

(i) finite, if $A \sim J_n$ for some n

(ii) infinite, if it is not finite.

(iii) countable or denumerable if $A \sim J$

(iv) atmost countable, if it is either finite or countable

(v) uncountable, if it is neither finite not countable

4.1.2 Note:

(1) The set Z of all integers is countable

(2) If E is countable and A is an infinite subset of E then A is countable

(3) The union of countable family of sets, each of which is countable also countable.

(4) The Cartesian product of two countable sets is countable.

(5) The set of rational numbers is countable.

(6) Every segment in R is uncountable

(7) The set of real numbers R is uncountable.

4.1.3 Definition : Let $\boldsymbol{X}$ be a non-empty set. For any $\boldsymbol{p}, \boldsymbol{q} \in \boldsymbol{X}$. We associate a real number $\boldsymbol{d}(\boldsymbol{p}, \boldsymbol{q})$ called the distance between $\boldsymbol{p}$ and $\boldsymbol{q}$ satisfying the following conditions.

(i) $\boldsymbol{d}(\boldsymbol{p}, \boldsymbol{q}) \geq \mathbf{0}$

(ii) $\boldsymbol{d}(\boldsymbol{p}, \boldsymbol{q}) = \mathbf{0}$ if and only if $\boldsymbol{p} = \mathbf{1}$

(iii) $\boldsymbol{d}(\boldsymbol{p}, \boldsymbol{q}) = \boldsymbol{d}(\boldsymbol{q}, \boldsymbol{p})$

(iv) $\boldsymbol{d}(\boldsymbol{p}, \boldsymbol{q}) \leq \boldsymbol{d}(\boldsymbol{p}, \boldsymbol{r}) + \boldsymbol{d}(\boldsymbol{r}, \boldsymbol{q})$ for any $\boldsymbol{r} \in \boldsymbol{X}$.

Then $\boldsymbol{d}(\boldsymbol{p}, \boldsymbol{q})$ is called a distance function or a metric. A set $\boldsymbol{X}$ on which a metric $\boldsymbol{d}$ is define is called a metric space and is denoted by $(\boldsymbol{X}, \boldsymbol{d})$.

4.1.4 Definition : Let $\boldsymbol{X}$ be a metric space

(i) A neighborhood of the point $\boldsymbol{b} \in \boldsymbol{X}$ is the set $\{\boldsymbol{q} \in \boldsymbol{X} \mid \boldsymbol{d}(\boldsymbol{p}, \boldsymbol{q}) < r\}$ and it is denoted by $\boldsymbol{N_n}(\boldsymbol{P})$.

(ii) Let $\boldsymbol{E} \subset \boldsymbol{X}$. A point $\boldsymbol{p} \in \boldsymbol{X}$ is a limit point of the set $\boldsymbol{E}$, if every neighborhood of $\boldsymbol{p}$ contains a point $\boldsymbol{q}$ such that $\boldsymbol{q} \in \boldsymbol{X}$ and $\boldsymbol{q} \neq \boldsymbol{p}$.

(iii) A set $\boldsymbol{E}$ is said to be closed if every limit point of $\boldsymbol{E}$ is a point of $\boldsymbol{E}$.

(iv) A point $\boldsymbol{p}$ is said to be an interior point of $\boldsymbol{E}$, if there is a neighborhood $\boldsymbol{N}$ of $\boldsymbol{p}$ such that $\boldsymbol{N} \subset \boldsymbol{E}$·

(v) A set of $\boldsymbol{E}$ is said to be open, if every point of $\boldsymbol{E}$ is an interior point of $\boldsymbol{E}$.

(vi) Every neighborhood is an open set.

4.1.5 Definition: Suppose $\boldsymbol{X}$ is a metric space and $\boldsymbol{E} \subset \boldsymbol{X}$. A collection of $\{\boldsymbol{G_\alpha}\}$ open sets in $\boldsymbol{X}$ is said to be an open cover, if $\boldsymbol{E} \subset \cup_\alpha \{\boldsymbol{G_\alpha}\}$.

4.1.6 Definition: A subset K of a metric space X is said to be compact, if every open cover of k contains a finite sub cover.

(i.e., there exists finite collection $\alpha_1, \alpha_2, \alpha_3, \ldots. \alpha_n$)

such that $k \subseteq G_{\alpha_1} \cup G_{\alpha_2} \cup G_{\alpha_3} \cup \ldots \cup G_{\alpha_n}$.

4.1.7 Note: (1) Every compact subset of a metric space is closed.

(2) Every closed subset of a compact metric space is compact.

4.1.8 Definition: Let X be a metric space and E is subset of X.

(1) The set E is said to be perfect set, if E is closed and if every point of E is a limit point of E.

(2) Two subsets A and B of X are said to be separated, if $A \cap \bar{B} = \emptyset$ and $\bar{A} \cap B = \emptyset$.

(3) The subset E of X is said to be connected, if it is not union of two non empty separated sets.

4.2 LIMITS OF FUNCTIONS:

4.2.1 Definition: Let (X, d_1) and (Y, d_2) be metric spaces; suppose $E \subseteq X$; f maps E into Y and P is a limit point of E. If there is a point $q \in Y$ with the property that for every $\epsilon > 0$, there exists a $\delta > 0$ such that $d_2(f(x), q) < \epsilon$ for all points $x \in E$ for which $0 < d_1(x, p) < \delta$, then we write $f(x) \to q$ as $x \to p$, or $\lim_{x\to\infty} f(x) = q$.

4.2.2 Note: Suppose $X = Y = \mathbb{R}$ and $d_1(x, y) = d_2(x, y) = |x - y|$ *for* all $X, Y \in \mathbb{R}$ and also suppose $E \subseteq \mathbb{R}$, P is a limit point of E. Then $f: E \to \mathbb{R}$ is said to have a limit as $x \to p$, if there is a $q \in \mathbb{R}$ satisfying the condition : for every $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - q| < \epsilon$ for all $x \in E$ with $0 < |x - p| < \delta$.

4.2.3 Example: Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by

$f(x) = \begin{cases} x + 2 & \text{if } x \neq 2 \\ 0 & \text{if } x = 2. \end{cases}$ Then $\lim_{x\to 2} f(x) = 4$

Let $\epsilon > 0$. Take $\delta = \epsilon$. Then for any x with $0 < |x - 2| < \delta$

$|f(x) - 4| = |x + 2 - 4| = |x - 2| < \delta = \epsilon$.

$\therefore \lim_{x\to 2} f(x) = 4$

4.2.4 Theorem: Let (X, d_1) and (Y, d_2) be metric spaces and $E \subseteq X$ and f maps E into Y and p is a limit point of E. Then $\lim_{x\to p} f(x) = q$ if and only if $\lim_{n\to\infty} f(p_n) = q$ for every sequences $\{p_n\}$ in E such that $P_n \neq P$ for all n and $\lim_{n\to\infty} f(p_n) = p$.

Proof: Given that X, Y are metric spaces and $E \subseteq X$ and f maps E into Y and P is limit point of E.

Suppose $\lim_{x\to p} f(x) - q$

Let $\{p_n\}$ be any sequence in E such that $P_n \neq P$ and $\lim_{n\to\infty} f(p_n) = p$

Let $\epsilon > 0$ Since $\lim_{x\to p} f(x) = q$, there exists $\delta > 0$ such that $d_2(t(x), q) < \epsilon$

if $x \in E$ and $0 < d_1(x,p) < \delta$ ....(1)

Since $P_n \neq P$ and $\lim_{n\to\infty} P_n = p$, there exists a positive integer N such that $0 < d_1(x,p) < \delta$ for all $n \geq N$

Then, by (1) $d_2(f(p_n), q) < E$ for all $n \geq N$

$\therefore \lim_{n\to p} f(n) = q$

Conversely suppose that $\lim_{x\to p} f(x) \neq q$

Now we will show that there exists a sequence $\{p_n\}$ of points in E such that $P_n \neq P$ and $\lim_{n\to\infty} p_n = p$ does not imply $\lim_{n\to\infty} p_n = q$

Since $\lim_{x\to p} f(x) \neq q$ there exists $\epsilon > 0$ such that for every $\delta > 0$, There exists a point $x \in E$ (depending on δ) with $d_1\ (f(x), q) \geq E$ but $0 < d_1(x; p) < \delta$. This implies for each $\delta_n - \frac{1}{n}(n - 1,2, \ldots.)$, there exists a point $p_n \in E$ such that $d_2\ (f(p_n), q) \geq E$ but $0 < d_p(p_n; p) < 0$, Consequently $\lim_{n\to\infty} P_n \neq p$

Now we will show that $P_n \neq P$ for all n and $\lim_{n\to\infty} P_n = p$

Since $0 < d_1(p_n; p) < \frac{1}{n}$ we have $P_n \neq P$ for $n = 1,2, \ldots.$

Let $\epsilon > 0$. Choose a positive integer N such that $\epsilon < \frac{1}{n}$. Now for all $n \geq N$. Consider $d_1(p_n; p) < \frac{1}{n} \leq \frac{1}{N} < \epsilon$

This implies $d_1(p_n; p) < \epsilon$ for all $n \geq N$ and hence $\lim_{n\to\infty} P_n = p$

Thus there exists a sequence $\{p_n\}$ of points in E such that $P_n \neq P$ and $\lim_{n\to\infty} P_n = p$ but $\lim_{n\to\infty} P_n \neq q$.

4.2.5 Corollary: Suppose f is mapping of a metric space (X, d_1) into a metric space (Y, d_2). If $\lim_{x\to p} f(x)$ exists in Y, then it is unique.

Proof: Suppose $\lim_{x\to p} f(x)$ exists in Y.

Suppose $\lim_{x\to p} f(x) = q_1$ and $\lim_{x\to p} f(x) = q_2$ where $q_1, q_2 \in Y$.

Claim: $q_1 = q_2$

Let $\{p_n\}$ be any sequence of points in X such that $P_n \neq P$ and $\lim_{n\to\infty} p_n = p$. Then by the above theorem, $\lim_{n\to\infty} f(p_n) = q_1$ and $\lim_{n\to\infty} f(p_n) = q_2$. So $\{f(p_n)\}$ is 0, sequence of points in Y such that , $\lim_{n\to\infty} f(p_n) = q_1$ and $\lim_{n\to\infty} f(p_n) = q_2$: since limit of a sequence is unique.

We have $q_1 = q_2$

4.2.6 Definition: Let X be a metric space and let f and g be complex valued functions defined on X. Now we define $f \pm g$, fg, f/g as follows.

Let $x \in X$. Define $(f \pm g)(x) = f(x) \pm g(x)$

$$(fg)(x) = f(x)g(x)$$

and $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$ $if\ g(x) = 0$

4.2.7 Definition: Let f and g be functions defined from metric space X into $\mathbb{R}^k$

Then we define

$(f \pm g)(x) = f(x) \pm g(x)$

$(fg)(x) = f(x)g(x)$ and

$(\lambda f)(x) = \lambda f(x)$ for any real λ and for all $x \in X$. If f and g are real valued functions and if $f(x) \geq g(x)$ for all $x \in X$ we write $f \geq g$.

4.2.8 Theorem: Suppose (X, d) is a metric space and f, g are complex valued functions defined on $E \subseteq X$. Suppose p is a limit point of E. If $\lim_{x\to p} f(x) = A$ and $\lim_{x\to p} g(x) = B$ then

(i) $\lim_{x\to p}(f + g)(x) = A + B$

(ii) $\lim_{x\to p}(fg)(x) = AB$

(iii) $\lim_{x\to p}\left(\frac{f}{g}\right)(x) = \frac{A}{B}$ provided $B \neq 0$

Proof: Since $\lim_{x\to p} f(x) = A$ by Theorem 4.2.4. We have $\lim_{n\to\infty} f(p_n) = A$ for any sequence $\{p_n\}$ of points in E with $\lim_{n\to\infty} p_n = p$ and $p_n \neq p$ for all n.

Since $\lim_{n\to p} g(x) = B$ by Theorem 4.2.4. We have $\lim_{n\to\infty} g(p_n) = B$ for any sequence $\{p_n\}$ of points in E with $\lim_{n\to\infty} p_n = p$ and $p_n \neq p$ for all n.

(i) Suppose that $\{p_n\}$ is a sequence of points in E such that $\lim_{n\to\infty} p_n = p$ and $p_n \neq p$ for all n. Consider

$$\lim_{n\to\infty}(f + g)(p_n) = \lim_{n\to\infty} f(p_n) + g(p_n)$$

$$= \lim_{n\to\infty} f(p_n) + \lim_{n\to\infty} g(p_n)$$

$$= A + B$$

Therefore, $\lim_{x\to p}(f+g)(x) = A+B$

(ii) Suppose that $\{p_n\}$ is a sequence of points in E such that $\lim_{n\to\infty} p_n = p$ and $p_n \neq p$ for all n. Consider

$$\lim_{n\to\infty}(fg)(p_n) - \lim_{n\to\infty}\left(f(p_n) + g(p_n)\right)$$

$$= \lim_{n\to\infty} f(p_n) + \lim_{n\to\infty} g(p_n)$$

$$= AB$$

Therefore, $\lim_{x\to p}(fg)(x) = AB$

(iii) Suppose that $\{p_n\}$ is a sequence of points in E such that $\lim_{n\to\infty} p_n = p$ and $p_n \neq p$ for all n. Consider

$$\lim_{n\to\infty}\left(\frac{f}{g}\right)(p_n) = \lim_{n\to\infty}\left(\frac{f(p_n)}{g(p_n)}\right)$$

$$= \frac{\lim_{n\to\infty} f(p_n)}{\lim_{n\to\infty} g(p_n)}$$

$$= \frac{A}{B}$$

Therefore, $\lim_{x\to p}\left(\frac{f}{g}\right)(x) = \frac{A}{B}$.

4.3 CONTINUOUS FUNCTIONS:

4.3.1 Definition: Suppose (X, d_1) and (Y, d_2) are metric spaces, $E \subseteq X$, $P \in E$ and f maps E into Y. Then f is said to be continuous at p if for every $\epsilon > 0$. There exists a $\delta > 0$ such that $d_1(f(x), f(p)) < \epsilon$ for all points $x \in E$ for which $d_1(x,p) < \delta$. If f is continuous at every point of E then f is said to be continuous on E.

4.3.2 Definition: Let (X, d) be a metric space and $E \subseteq X$: A point $p \in E$ is said to be an isolated point of E if there is a neighborhood $N_\delta(p)$ of P such that $N_\delta(p)$ has just one point p of the set E.

That is $N(p) = \{x \in E \mid d(x,p) < \delta\} = \{p\}$ and $\{x \in E \mid 0 < d(x,p) < \delta\} = \emptyset$

Therefore if p is an isolated point of E, then the condition, in definition 4.3.1, $d_2(f(x), f(p)) < \epsilon$ for all $x \in E$ with $d_1(x,p) < \delta$ holds obviously. Hence if $p \in E$ is an isolated point of E. Then f is continuous at p.

4.3.3 Example: Define $f: \mathbb{R} \rightarrow \mathbb{R}$ as

$$f(x) = \begin{cases} x+2 & \text{if} \quad x \neq 2 \\ 0 & \text{if} \quad x = 2. \end{cases}$$

Then $\lim_{x \to 2} f(x) = 4$. But $f(2)$. So f is continuous at $x = 2$.

4.3.4 Example: Define $f: \mathbb{R} \rightarrow \mathbb{R}$ as $f(x) = x + 2$ for all $x \in \mathbb{R}$. Then $\lim_{x \to 2} f(x) = 4$ and is equal to $f(2)$. So f *is* continuous at $x = 2$.

4.3.5 Theorem: Let (X, d_1) and (Y, d_2) be metric spaces, $E \subseteq X$, and f maps E into Y. If $p \in E$ is a limit point of E, then f is continuous at p if and only if

$$\lim_{x \to p} f(x) - f(p)$$

Proof: Consider f *is* continuous at p if and only if for each $\epsilon > 0$, there exists a $\delta > 0$ such that $d_2(f(x), f(p)) < \epsilon$ for all points $x \in E$ for which $d(x, p) < \delta$ if and only if $\lim_{x \to p} f(x) = f(p)$ ($\because p$ is a limit point of E).

4.3.6 Theorem: Suppose (X, d_1), (Y, d_2) and (Z, d_3) are metric spaces, $E \subseteq X$, f maps E into Y, g maps the range of f, $f(E)$, into Z and h is the mapping of E into Z defined by $h(x) = g(f(x))$ for all $x \in E$. If f is continuous at a point $p \in E$ and if g is continuous at the point $f(p)$, then h is continuous at p.

Proof: Suppose f *is* continuous at $p \in E$ and g is continuous at the point $f(p)$. Let $\epsilon > 0$. Since g is continuous at $f(P)$, there exists an $\eta > 0$ such that

$d_3(g(y), g(f(p))) < \epsilon$ whenever $d_2(y, f(p)) < \eta$ and $y \in f(E)$……*(1)*

Since f *is* continuous at p, there exists a $\delta > 0$ such that $d_2(f(x), f(p)) < \eta$

whenever $d_1(x, p) < \delta$ and $x \in E$…….*(2).*

Suppose $x \in E$ such that $d_1(x, p) < \delta$. Then consider

$d_3(h(x), h(p)) = d_3(g(f(x)), g(f(p))) < \epsilon$ *(from (1) and (2))*

Thus for $\epsilon > 0$, there exists $\delta > 0$ such that

$d_3(h(x), h(P)) < \epsilon$ whenever $d_1(x, p) < \delta$. Therefore h is continuous at p.

In the above theorem, h is called the composition f and g and we write $h = g o f$.

4.3.7 Theorem: Suppose (X,d) is a metric space and f, g are complex valued functions defined on X. If f and g are both continuous at $p \in X$, then $f+g$, fg and $\frac{f}{g}$ (if $g(p) \neq 0$) are continuous at $p \in X$.

Proof: Suppose (X, d) is a metric space and f,g are complex valued functions defined on X is continuous at p. So $f+g$, fg, and $\frac{f}{g}$ are continuous at p

Case (ii): Suppose p is a limit point of X.

By Theorem 4.3.5, f *is* continuous at p if and only if $\lim_{x\to p} f(x) = f(p)$ and g is x continuous at p if and only if

$\lim_{x\to p} g(x) = g(p)$. Then by Theorem 4.2.8

$\lim_{x\to p}(f+g)(x) = f(p) + g(p) = (f+g)(p)$

$\therefore f+g$ is continuous at p. (By Theorem 4.3.5)

Consider $\lim_{x\to p}(fg)(x) = f(p).g(p) = (fg)(p)$ (By Theorem 4.2.8)

By Theorem 4.3.5, fg is continuous at p. Suppose $g(p) \neq 0$

Consider $\lim_{x\to p}\left(\frac{f}{g}\right)(x) = \frac{f(p)}{g(p)}$ (By Theorem 4.2.8)

By Theorem 4.3.5, $\frac{f}{g}$ is continuous at p.

4.3.8 Theorem: (a) : Let f_1, f_2, f_3, f_4 be real functions on a metric space X, and let f be the mapping of X into $\mathbb{R}^k$ defined by $f(x) = (f_1(x), f_2(x), \ldots\ldots, f_k(x))$ $(x \in X)$; then f is continuous if and only if each of the functions $f_1, f_2, \ldots\ldots f_k$ are continuous.

(b): If f and g are continuous mappings of X into $\mathbb{R}^k$ then $f+g$ and $f.g$ are continuous on X.

Proof: Given that f is mapping of a metric space (X,d) into $\mathbb{R}^k$ defined by $f(x) = (f_1(x), f_2(x), \ldots\ldots, f_k(x))$ where $f_1, f_2, \ldots\ldots, f_k$ are real valued functions defined on X.

(a) : *Assume* f *is* continuous *on* X.

Let $x \in X$ and let $\epsilon > 0$. Since f *is* continuous at x . Then there exists a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $d(x,y) < \delta$, for $y \in X$

$\Rightarrow \left(\sum_{i=1}^{k} |f_i(x) - f_i(y)|^2\right)^{\frac{1}{2}} < \epsilon$ for $d(x,y) < \delta$

$\Rightarrow |f_i(x) - f_i(y)| < \epsilon$ for $d(x,y) < \delta$ and for $1 < i < k$

$\Rightarrow f_i$ is continuous at x for $1 \leq i \leq n$

Since $x \in X$ is arbitrary, f_i is continuous on $1 \leq i \leq k$.

Now, we will show that f is continuous on X.

Let $x \in X$ and let $\epsilon > 0$. Since each f_i is continuous at x, there exists a $\delta_i > 0$.

Such that $|f_i(x) - f_i(y)| < \frac{\epsilon}{\sqrt{k}}$ whenever $d(x,y) < \delta_i$ and for $1 < i < k$.

Take $\delta = \min\{\delta_1, \delta_2, \dots, \delta_k\}$.

Suppose $d(x,y) < \delta$. Then $d(x,y) < \delta$ for $1 < i < k$.

$\Rightarrow |f_i(x) - f_i(y)| < \frac{\epsilon}{\sqrt{k}}$ for $1 \leq i \leq k$.

Consider $|f_i(x) - f_i(y)|^2 = (\sum_{i=1}^{k} |f_i(x) - f_i(y)|^2)^{\frac{1}{2}} < k.\frac{\epsilon}{k} - c^2$

$\Rightarrow |f(x) - f(y)| < \epsilon$

Therefore f is continuous at x.

Since $x \in X$ is arbitrary, f is continuous on X.

(b): Suppose f and g are continuous mappings of X into defined by $f(x) = (f_1(x), f_2(x), \dots, f_k(x))$ and $g(x) = (g_1(x), g_2(x), \dots, g_k(x))$ with $f_1, f_2, \dots, f_k$; $g_1, g_2, \dots, g_k$ are real valued functions defined on X. Since f and g are continuous on X, by **(a),** each f_i is continuous on x and each g_i is continuous on X. Then by Theorem 4.3.7. $f_i + g_i$ and $f_i g_i$ are continuous on X for $1 \leq j \leq k$. Since $(f + g)(x) = (f_i + g_1)(x).(f_2 + g_2)(x) \dots (f_k + g_k)(x)$ for all $x \in X$, by **(a)**, $f + g$ is continuous on x for $1 \leq j \leq k$. Since $f_i g_i$ is continuous on X for $1 \leq j \leq k$ we have $\sum_{i=1}^{k} f_i g_i$ is continuous on X and hence $f.g$ is continuous on X.

4.3.9 Example: Every polynomial with complex coefficients is continuous at every point of ℂ. For, let $p(x) = \alpha_0 + \alpha_1 x + \dots + \alpha_n x^n$ where $\alpha_0 + \alpha_1 + \dots + \alpha_n$ are complex numbers.

Consider p: ℂ → ℂ as a function.

Define $I: p:$ ℂ → ℂ as $I(x) = x$ for all $x \in$ ℂ. Then I is continuous at every point of ℂ for $\epsilon > 0$ is given, taking $\delta = \epsilon$, for all $x \in$ ℂ with $0 < |x - a| < \delta$ we have $|I(x) - I(a)| = |x - a| < \delta = \epsilon \Rightarrow I$ is continuous.

$\Rightarrow I^2(x) = I(x)I(x) = x^2$ is continuous

……………………………………………

……………………………………………

$I^n(x) = x^n$ is continuous

It is easy to verify that every constant function is continuous.

Therefore $f_0(x) = \alpha_0$

$f_1(x) = \alpha_1 x = \alpha_1 I(x)$

$f_2(x) = \alpha_2 x = \alpha_2 I^2(x)$

……………………..

……………………

$f_n(x) = \alpha_n x = \alpha_n I^n(x)$ are all continuous on $\mathbb{C}$.

Hence $f_0(x) + f_1(x) + \ldots\ldots\ldots\ldots + f_n(x) = p(x)$ is continuous on $\mathbb{C}$.

4.3.10 Definition: Suppose $f: A \to B$ is a mapping where A and B are any two sets. For any $T \subset A$, $f(T) = \{f(x) | x \in T\}$ is called the image of T under f For any $V \subset B$, the set $\{x \in A | f(x) \in V\}$ is called the inverse image of V under I and is denoted by $f^{-1}(V)$. That is $f^{-1}(V) = \{x \in A | f(x) \in V\}$.

4.3.11 Theorem: Suppose $f: A \to B$ is a mapping. Then for every set $V \subseteq B$,

(i) $f^{-1}(V^c) = \left[f^{-1}(V)\right]^c$

(ii) $f\left(f^{-1}(V)\right) \subseteq V$

Proof: (i) Consider $x \in f^{-1}(V^c) \Leftrightarrow x \in A$ and $f(x) \in V^c$

$\Leftrightarrow x \in A$ and $f(x) \notin V \Leftrightarrow x \in f^{-1}(V)$

$\Leftrightarrow x \in \left(f^{-1}(V)\right)^c$

$$f^{-1}(V^c) = \left(f^{-1}(V)\right)^c$$

(ii) Suppose $t \in f\left(f^{-1}(V)^c\right) = \left\{f(x) \mid x \in f^{-1}(V)\right\}$

$\Rightarrow t = f(x)$ for some $x_0 \in f^{-1}(V)$

$\Rightarrow t = f(x)$ for some $x_0 \in A$ with $f(x_0) \in V$

$\Rightarrow t \in V$

$\therefore f\left(f^{-1}(V)\right) \subseteq V$.

4.3.12 Theorem: A mapping f of a metric space (X, d_1) into a metric space (Y, d_2) is continuous on X if and only if. $f^{-1}(V)$ is open in X for every open set V in Y.

Proof: let $f: X \to Y$ be a mapping

Suppose f is continuous on X.

Let V be an open set in Y.

Now we will show that every point in $f^{-1}(V)$ is an interior point of it.

Let $x \in f^{-1}(V)$. *Then* $f(x) \in V$. since V is an open set in Y, there exists $\epsilon > 0$ such that $N_\epsilon(f(x)) \in V$. Since f is continuous on X. f is continuous at x. Then there exists $\delta > 0$ such that $d_2(f(z), f(x)) < \epsilon$ whenever $d_1(z, x) < \delta$

This implies $f(z) \in N_\epsilon(f(x))$ whenever $z \in N_\delta(x)$. That is, $f(z) \in V$ whenever $z \in N_\delta(x)$. That is $z \in f^{-1}(V)$ whenever $z \in N_\delta(x)$ and hence $N_\delta(x) \subseteq f^{-1}(V)$. Thus $f^{-1}(V)$ is open in X whenever V is open in X.

$\therefore$ x is an interior point of $f^{-1}(V)$. Hence $f^{-1}(V)$ is open in X.

Thus $f^{-1}(V)$ is open in X whenever V is open in Y.

Conversely suppose that $f^{-1}(V)$ is open in X for every open set V in Y.

Now we will show that f is continuous at ever point of let $p \in X$ and let $\epsilon > 0$ now

$N_\epsilon(f(p))$ is an open set in Y. By our supposition $f^{-1}(N(f(p)))$ is an open set in X and $p \subset f^{-1}(N(f(p)))$. Then there exists $\delta > 0$ such that

$N_\delta(p) \subseteq f^{-1}(N_\epsilon(f(p)))$. This implies $f(N_\delta(p)) \subseteq N_\epsilon(f(p))$

That is, if $d_1(x, p) < \delta$, then $d_2(f(x), f(p)) < \epsilon$. This shows that f *is* continuous at p. Since $p \in X$ is arbitrary. f is continuous on X.

Thus f *is* continuous on X if and only if $f^{-1}(V)$ is open in X whenever V is open in X.

4.3.13 Corollary: A mapping of a metric space X into a metric space Y is continuous if and only if $f^{-1}(V)$ is closed in X for every closed set V in Y.

Proof: let $f: X \to Y$ be a function. Let V *be* any closed set in Y. Consider f is continuous on X if and only if $f^{-1}(V^c)$ is open in X (by Theorem 4.3.12) if and only if $\left(f^{-1}(V)\right)^c$ is open in X ($\because f^{-1}(V)^c = f^{-1}(V)$ if and only if $f^{-1}(V)$ is closed in X). Thus f is continuous on X if and only if $f^{-1}(V)$ is closed in X for every closed set V in Y.

4.3.14 Problem: If f is a continuous mapping of a metric space X into a metric space Y, prove that $f(\bar{E}) \subseteq \overline{f(E)}$ for any subset E of X.

Solution: Suppose f *is* a continuous mapping of a metric space X into a metric space Y and $E \subseteq X$. Now $\overline{f(E)}$ is a closed subset of Y containing $f(E)$. Since *f* is continuous on *X*, by corollary 4.3.13, $f^{-1}\overline{f(E)}$ is a closed set in *X* and $E \subseteq f^{-1}\overline{f(E)}$. Since *E* is the smallest closed set containing *E*. we have $\bar{E} \subseteq f^{-1}\overline{f(E)}$. This implies $f(\bar{E}) \subseteq \overline{f(E)}$. Thus for any sub set *E* of *X*, $f(\bar{E}) \subseteq \overline{f(E)}$.

4.3.15 Problem: Let f be a continuous real function on a metric space X. Let $Z(f)$ (the zero set of f be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.

Solution: Given that f *is* a continuous real function on a metric space X and $Z(f) = \{p \in X | f(P): 0\}$.

Claim: $Z(f)$ is a closed set.

Let *y* be a limit point of $Z(f)$ in *X*. Then by a known theorem, there exists a sequence $\{x_n\}$ of points in $Z(f)$ such that $x_n \to y$. Since f is continuous, by Theorem 4.2.4, and Theorem 4.3.12. we have $f(x_n)$ converges to $f(y)$. This implies $f(y) = \lim_n f(x_n) = 0$ ($\therefore x_n \in Z(f)$ for all n) and hence $y \in Z(y)$, This shows that $Z(f)$ is a closed set in *X*.

4.3.16 Problem: Let f and g be continuous mappings of a metric space X into a metric space Y and let E be a dense subset of X. Prove $f(E)$ is dense in $f(X)$. If $g(P) = f(P)$ for all $p \in X$, prove that $g(P) = f(P)$ *for* all $p \in X$ (In other words, a continuous mapping is determined by its values on a dense subset of its domain)

Proof: Given that f and g are continuous mappings of a metric space X into a metric space Y and E is a dense subset of X,

Claim: $f(E)$ is dense in $f(X)$. That is, $\overline{f(E)} = f(X)$. Clearly $\overline{f(E)} \subset f(X)$

Let $y \subset f(X)$. If $y \subset f(E)$, then $y \subset \overline{f(E)}$

Suppose $y \notin \overline{f(E)}$ in this case we will show that, y is a limit point of $f(E)$.

Since $y \in f(E)$, $y = f(x)$ for some $x \in X$. Then $x \notin E$.

Since E is dense in X, x is a limit point of E. Then by a known result, there exists a sequence $\{x_n\}$ of points in E such that $\{x_n\}$ converges to x. Since f is continuous and $\{x_n\}$ converges x, by a known result. $\{f(x_n)\}$ converges to $f(x)$. Now $\{f(x_n)\}$ is a sequence of points in

$f(E)$ such that $\{f(x_n)\}$ converges to y. This implies $y \in \overline{f(E)}$ this shows that $f(x) \subset \overline{f(E)}$ and hence $\overline{f(E)} = f(x)$.

Suppose $f(P) = g(P)$ for all $p \in X$.

Now we will show that $f(x) = g(x)$ for all $x \in X$

Let $x \in X$. Since E is dense in X, there exists a sequence $\{x_n\}$ of points in E such that $\{x_n\}$ converges to x Since f and g are continuous on X, we have $\{f(x_n)\}$ converges to $f(x)$ and $\{g(x_n)\}$ converges to $g(x)$.

Consider $f(x) = \lim_n f(x_n) = \lim_n g(x_n) = g(x)$

$(\because x_n \in E$ for all n and $f(x_n) = g(x_n))$

$\therefore f(x) = g(x)$ for all $x \in X$.

4.3.17 Problem: The function $f: \mathbb{R}^k \to \mathbb{R}^k$ given by $f(x) = x^2$ is continuous on $\mathbb{R}^k$.

Solution: Let $x_0 \in \mathbb{R}^k$ and $\epsilon > 0$

Now $|x^2 - x_0^2| = |(x + x_0)(x - x_0) \leq (|x| + |x_0|)||x - x_0|)$.

If $|x - x_0| < 1$ so that $|x| < |x_0| + 1$

Then $|x^2 - x_0^2| < (2|x_0| + |x - x_0|) < \epsilon$

If $|x - x_0| < \frac{\epsilon}{2|x_0|+1}$

So if $\delta < \min\left(1, \frac{\epsilon}{2|x_0|+1}\right)$ then $|x - x_0| < \delta$

$\Rightarrow |x^2 - x_0^2| < \epsilon$, so f is continuous on R.

Model Examination Questions

1. If (X, d_1) and (Y, d_2) are metric spaces and $E \subseteq X$ and if f maps E into Y and p is a limit point of E, then show that $\lim_{x\to p} f(x) = q$ if and only if $\lim_{n\to\infty} f(p_n) = q$ for every sequence $\{p_n\}$ in E such that $p_n \neq p$ for all n and $\lim_{n\to\infty} p_n = p$.

2. Suppose X, Y and Z are metric spaces and f maps X into Y and g maps Y into Z and h is the mapping of X into Z, defined by $h(x) = g(f(x))$ for all $x \in X$. If f is continuous on X and g is continuous on Y, then show that h is continuous from X into Z.

3. Show that a mapping f of a metric space X into a metric space Y is continuous on X if and only if $f^{-1}(V)$ is open in X for every open set V in X for every open set V in Y.

4. Let f and g be continuous mappings of a metric space X into a metric space Y and let E be a dense subset of X. Prove that $f(E)$ is dense in $f(X)$. If $g(p) = f(P)$ for all $p \in E$. then prove that $g(P) = f(P)$ for all $p \in X$.

Exercises

1. Suppose f is a real function defined on $\mathbb{R}$ which satisfies

$\lim_{h\to 0}[f(x+h) - f(x-h)] = 0$ for every $x \in \mathbb{R}$. Does this imply that f is continuous?

2. If f is a real continuous function defined on a closed set $E \subset \mathbb{R}$. Prove that there exist continuous real functions g on $\mathbb{R}$ such that $g(x) = f(x)\cdot$ for all $x \in E$.

Answers to Self Assessment Questions

For **1** see definition **4.3.1**

For **2**, see example **4.3.3**

For **3**, see problem **4.3.15**

4.4 SUMMARY:

This lesson covers the fundamental concepts of limits and continuity of functions on metric spaces, essential for modeling real-world applications such as motion, force, and energy. The lesson aims to develop problem-solving skills in calculus and analysis. And also covers Definitions of key concepts, Proofs of relevant theorems, Corollaries to reinforce understanding, and Practice problems to develop problem-solving skills.

4.5 TECHNICAL TERMS:

- Compact
- Complex valued function
- Connected
- Continuous
- Countable **or** Denumerable
- Finite
- Function
- Infinite
- Interior point
- Isolated point

- Metric spaces
- Neighbourhood
- Open cover
- Perfect set
- Polynomial
- Range
- Real valued functions
- Subset
- Uncountable
- Union

4.6 SELF ASSESSMENT QUESTIONS:

1. When do you say that a function f from a metric space into a metric space is continuous?

2. Show that the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f\{x\} = x + 2$ for all $x \in \mathbb{R}$ is continuous at $x = 2$.

3. Let f be a continuous real function on a metric space X. Let $Z(f)$ be the set of all $p \in X$ at which $f(p) = 0$. Show that $Z(f)$ is closed.

4.7 SUGGESTED READINGS:

1. Principles of Mathematical Analysis, Third Edition, Mc Graw-HillInternational Editions Walter Rudin.
2. Mathematical Analysis by Tom M. Apostol, Narosa Publishing House, 2nd Edition, 1985.

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LESSON-5

CONTINUITY, COMPACTNESS AND CONNECTEDNESS

OBJECTIVES:

The objective of the lesson is to help the learners to understand

- ❖ To understand the concepts of continuity, compactness and connectedness.
- ❖ To develop problem solving skills using continuity, compactness and connectedness.

STRUCTURE:

5.0 INTRODUCTION:

In this lesson the behaviour of continuous functions-when they are defined on compact sets or connected sets is discussed. It is proved that if f is a continuous mapping of a compact metric space X into a metric space Y, then $f(x)$ is Compact. It has also been proved that a continuous 1–1 mapping of a compact metric space onto a metric space is a homomorphism. Further the uniform continuity of a function from a metric space into another metric space is defined. It is also proved that a continuous mapping of a compact metric space into a metric space is uniformly continuous Further it is proved that continuous image of a connected set is connected.

5.1 CONTINUITY AND COMPACTNESS:

5.1.1 Definition: A mapping f of a metric space X into $\mathbb{R}^k$ is said to be bounded if there exists a real number M such that $|f(x)| \leq M$ for all $x \in X$. That is $f: X \to \mathbb{R}^k$ is bounded if the image $f(X)$ is a bounded set in $\mathbb{R}^k$.

5.1.2 Theorem: Suppose f *is* a continuous mapping of a compact metric space X into a metric space Y. Then $f(X)$ is compact.

Proof: Suppose X, is a compact metric space and $f: X \to Y$ is a continuous mapping. Let $\{V_\alpha\}_{\alpha \in \Delta}$ be an open cover of $f(X)$ in Y. Then $f(X) \subseteq \bigcup_{\alpha \in \Delta} V_\alpha$. Since f is continuous on X and V_α is open in Y for each $\alpha \in \Delta$ the inverse image $f^{-1}(V_\alpha)$ is open in X for each $\alpha \in \Delta$. Also it is clear that

$X \subseteq \bigcup_{\alpha\in\Delta} f^{-1}(V_\alpha)$. This implies that $\left(f^{-1}(V_\alpha)\right)_{\alpha\subset\Delta}$ is an open cover for X. Since X is compact, there exists $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Delta$ such that $X \subseteq \bigcup_{\alpha\in\Delta}^{n} f^{-1}\left(V_{\alpha_i}\right)$. This implies$f(X) \subseteq \bigcup_{\alpha\in\Delta}^{n} V_{\alpha_i}$. Therefore $f(X)$ is compact. This theorem can also be stated as "The image of a compact metric space under a continuous mapping is a compact metric space or the continuous image of a compact metric space is compact".

5.1.3 Theorem: If f is a continuous mapping of a compact metric space X into then $f(X)$ is closed and bounded. Thus f is bounded.

Proof: Suppose f *is* a continuous mapping of a compact metric space X into. Then by Theorem 5.1.2, $f(x)$ is a compact sub set of $\mathbb{R}^k$. Since every compact subset of $\mathbb{R}^k$ is closed and bounded, $f(X)$ is closed and bounded. This implies there exists a real number M such that $|f(x)| \leq M$ for all $x \in X$. Therefore f is bounded.

5.1.4 Corollary: If X is a compact metric space and f is a continuous real valued function on X, then $f(X)$ is bounded.

Proof: Taking $k = 1$, the corollary follows.

5.1.5 Theorem: Suppose f is a continuous real function on a compact metric space X and $\sup_{p\in X} f(p)$, $\inf_{p\in X} f(p)$. Then there exist points $p, q \in X$ such that $f(P) = M$ and $f(q) = m$.

Proof: Let X be a compact metric space and f be a continuous real function on X. Then by Theorem 5.1.3, $f(X)$ is closed and bounded. Since $f(X)$ is bounded, we have $\sup f(x)$ and $\inf f(x)$ exist in $\mathbb{R}$. Since $f(X)$ is closed in $\mathbb{R}$, by a known theorem, $\sup f(x) \in f(X)$ and $\inf f(x) \in f(X)$. This implies $\sup_{x\in X} f(X) = f(p)$ and $\inf_{x\in X} f(X) = f(p)$, for some $p, q \in X$. Thus there exist $p, q \in X$ such that $f(P) = M$ and $f(q) = m$.

5.1.6 Note: The notation in the above theorem means that M is the least upper bound of the set of all numbers $f(p)$, where p ranges over X and that m is the greatest lower bound of this set of numbers.

5.1.7 Note: The conclusion in the above theorem may also be stated as follows. There exist points p and q in X such that $f(q) \leq f(x) \leq f(p)$ for all $x \in X$. that is, f attains its maximum *(at p)* and minimum (at q).

5.1.8 Theorem: Suppose f is a continuous 1-1 mapping of a compact metric space X onto a metric space Y. Then the inverse mapping f^{-1} defined on Y by $f^{-1}\left(f(x)\right) = x(x \in X)$ is a continuous mapping of Y onto X.

Proof: Suppose f is a continuous 1-1 mapping of a compact metric space X onto a metric space Y. To show f^{-1} is continuous, by Theorem 4.3.13, it is enough, if we show that $f(V)$ is open in Y for every open set V in X. Let V be any open set in X. Then V is a closed subset of X. Since every closed subset of a compact metric space is compact, we have V^k is a compact subset of X.

Since f is continuous on X, by Theorem 5.1.2 $f(V)^c$ is a compact subset of Y. Since every compact subset of a metric space is closed, we have $f(V)$ is closed in Y. Since f is 1-1 and onto, $f(V) = (J(V)^c)^c$. This implies $f(V)$ is open in Y. Thus f^{-1} is continuous.

5.1.9 Definition: A one-one, onto function f of a metric space X onto a metric space Y is said to be a homomorphism if both f and f^{-1} are continuous.

5.1.10 Note: By theorem 5.1.8, a one-one, onto continuous function f on a compact metric space is always a homomorphism.

5.1.11 Definition: Let f be a mapping of a metric space (X, d_1) into a metric space (Y, d_2). We say that f is uniformly continuous on X; if f or every $\epsilon > 0$ there exists a $\delta > 0$ such that $d_2(f(p), f(q)) < \delta$ for all p and q in X for which $d_1(p, q) < \delta$.

5.1.12 Example: Define $f: \mathbb{R} \to \mathbb{R}$ as $f(x) = 2x$ for all $x \in \mathbb{R}$. Then f is uniformly continuous

For, let $\epsilon > 0$. Take $\delta = \frac{\epsilon}{2}$, suppose $x, y \in \mathbb{R}$ such that $|x - y| < \delta$.

Consider $|f(x) - f(y)| = |2x - 2y| = 2|x - y| < 2\delta = 2.\frac{\epsilon}{2} = \epsilon$.

Which implies, if $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$.

$\therefore f$ is continuous.

5.1.13 Note: Every uniformly continuous function is continuous but the converse need not to be true.

Proof: For, suppose f is a uniformly continuous function from a metric space (X, d_1) into a metric space (Y, d_2). Let $\epsilon > 0$. Since f is uniformly continuous on X; there exists a $\delta > 0$ such that $d_2(f(x), f(y)) < \epsilon$ whenever $d_1(x, y) < \delta$...............*(1)*

Let $x \in X$. Let $y \in X$ such that $d_1(x, y) < \delta$. Then by (1), $d_2(f(x), f(y)) < \epsilon$.

Therefore f is continuous at x. Since $x \in X$ is arbitrary, we have f is continuous on X. Thus every uniformly continuous function is continuous. In general the converse is not true. For, consider the following example.

5.1.14 Example: Define $f(0,1) \to \mathbb{R}$ as $f(x) = \frac{1}{x}$ for all $x \in (0,1)$. First we show that f is continuous x

Let $\epsilon > 0$ and $x \in (0,1)$. Choose a $\delta > 0$. Such that $\delta < \frac{\epsilon x^2}{1+\epsilon x}$

Consider $\delta < \frac{\epsilon x^2}{1+\epsilon x} \Leftrightarrow \delta(1 + \epsilon x) < \epsilon x^2 \Leftrightarrow \delta < \epsilon x^2 - \delta\epsilon x$

$\Leftrightarrow \delta < \epsilon(x - \delta)x \Leftrightarrow \frac{\delta}{x(x-\delta)} < \epsilon$.......................................*(1)*

Suppose $y \in (0,1)$ such that $|x - y| < \delta$. Then $x - \delta < y < x + \delta$.

$\Leftrightarrow \frac{1}{x+\delta} < \frac{1}{y} < \frac{1}{x-\delta}$ ....*(2)*

Consider $f(x) - f(y) = \left|\frac{1}{x} + \frac{1}{y}\right| = \left|\frac{y-x}{xy}\right| = \frac{|y-x|}{xy} < \frac{\delta}{xy}$

$< \frac{\delta}{x(x-\delta)} < \epsilon$ (by *(1)* and *(2)*)

This shows that f is continuous at x and hence f *is* continuous on $(0,1)$.

Now we will show that f is not uniformly continuous on $(0,1)$.

Then for $\epsilon = 1$, there exists $\delta > 0$ such that $|f(x) - f(y)| < 1$ whenever $|x - y| < \delta$............(3)

Since $\delta > 0$, there exists a positive integer N such that $\frac{1}{N} < \delta$. Consider

$$\left|\frac{1}{N} - \frac{1}{N+1}\right| = \left|\frac{1}{N(N+1)}\right| < \frac{1}{N} < \delta$$

Now, $\frac{1}{N}, \frac{1}{N+1} \in (0,1)$, such that $\left|\frac{1}{N} - \frac{1}{N+1}\right| < \delta$

Then by *(3)*, $\left|f\left(\frac{1}{N}\right) - f\left(\frac{1}{N+1}\right)\right| < 1$, a contradiction.

So, f is not uniformly continuous.

Thus f is a continuous function but not uniformly continuous.

5.1.15 Theorem: Let f be a continuous mapping of a compact metric space (X, d_1) into a metric space (Y, d_2). Then f is uniformly continuous on X.

Proof: Given that f *is* a continuous mapping of a compact metric space X into a metric space Y.

Let $\epsilon > 0$. Since f is continuous on $q \in X$, for each $p \in X$, there exists a positive number δ_p, such that $q \in X$ with $d_1(p,q)$

Write $V_p = \left\{q \subset X \mid d_1(p,q) < \frac{\delta_p}{2}\right\}$. Then V_p is a neighbourhood of p and hence an open subset of X

Now $dS = \{V_p | p \in X\}$ is a class of open sets in X. It is dear that dS is an open cover for X.

Since X is compact, there exists $P_1, P_1, \ldots, P_n X$ such that $X = \cup_{i=1}^{n} V_{p_i}$........*(1)*

Take $\epsilon = \frac{1}{2} min\{\delta_{p_1}, \delta_{p_2}, \ldots . \delta_{p_n}\}$. Then $\delta > 0$.

Now let $p, q \in X$ be such that $d_1(p,q) < \delta$. By *(1)* there exists an integer m with $1 \leq m \leq n$.

such that $p \in V_{p_m}$. This implies $d_1(p, p_m) < \frac{\delta_{p_m}}{2}$.

Also, $d_1(q, P_m) < d_1(p,q) + d_2(P, P_m) < \delta + \frac{\delta_{p_m}}{2} \leq \delta_{p_m}$.

Then $d_2(f(p), f(P_m)) < \frac{\epsilon}{2}$ and $d_2(f(q), f(P_m)) < \frac{\epsilon}{2}$.

Consider $d_2(f(p), f(q)) < d_2(f(P), f(P_m)) + d_2(f(P_m), f(q))$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$$\Rightarrow d_2(f(p), f(q)) < \varepsilon.$$

This shows that f is uniformly continuous on X.

5.1.16 Theorem: Let E be a non-compact set in $\mathbb{R}$. Then

5.1.16.1 There exists a continuous function on E which is not bounded.

5.1.16.2 There exists a continuous and bounded function on E which has no maximum. If, in addition, E is bounded, then

5.1.16.3 There exists a continuous function on E which is not uniformly continuous.

Proof: Given that E is a non-compact subset of $\mathbb{R}$ .Since E is a non-compact subset of $\mathbb{R}$,then either E is bounded and E is not closed or E is closed and E is not bounded or E is not closed and not bounded.

Case (i) : Suppose E is bounded and E is not closed. Since E is not closed, there exists a point $x_0 \in \mathbb{R}$ such that x_0 is a limit point of E and $x_0 \notin E$.

Define $f: E \to \mathbb{R}$ as $f(x) = \frac{x}{x - x_0}$ for all $x \in E$.

Then f is continuous on E.

Now we will show that f is not bounded. That is, $f(E)$ is not bounded. Since x_0 is a limit point of E, there exists a sequence $\{x_n\}$ of points in E such that $x_n \to x_0$ as $n \to \infty$. This implies $x_n - x_0$ as $n \to \infty$ and and consequently $\frac{1}{x_n - x_0} \to \infty$ as $n \to \infty$.

Let $M > 0$. Since $\frac{1}{x_n - x_0} \to \infty$ as $n \to \infty$, there exists a positive integer N such that $\frac{1}{x_n - x_0} \to M$ for all $n \geq N$. This, implies $f(x_n) > M$ for all $n \geq N$. Therefore $f(E)$ is not bounded; **i.e**, f is not bounded.

Next we will show that f *is* not uniformly continuous on X. First, we show that $f(N_\delta(x_0) \cap E)$ is not bounded for all $\delta > 0$: Let $\delta > 0$ be any real number.

It is clear that $N_\delta(x_0) \cap E$ is bounded. Now we will show that x_0 is a limit point of $N_\delta(x_0) \cap E$. Let $r > 0$. Put $\eta = \min\{r, \delta\}$

Consider $N_\eta(x_0) \cap (E \cap N_\delta(x_0)) \backslash \{x_0\} = N_\eta(x_0) \cap \{x_0\} \neq \emptyset$

($\because$ x_0 is a limit point of E)

This implies that $N_r(x_0) \cap \{x_0\}(E \cap N_\delta(x_0)) \backslash \{x_0\} \neq \emptyset$ and hence limit point of $N_\delta(x_0) \cap E$.

Since $x_0 \notin E$, we have $x_0 \notin E \cap N_\delta(x_0)$. So $E \cap N_\delta(x_0)$ is a bounded set and x_0 is a limit point of $N_\delta(x_0) \cap E$ such that $x_0 \notin N_\delta(x_0) \cap E$. Therefore by the above argument, $f(N_\delta(x_0) \cap E)$ is not bounded. Since, $\delta > 0$ is arbitrary $f(N_\delta(x_0) \cap E)$ is not bounded for all $\delta > 0$.

Let $\epsilon > 0$ and $\delta > 0$. Let $x \in N_\delta(x_0) \cap E$ .Then $x \in E$ and $|x - x_0| < \delta$ and $|x - x_0| > 0$ ($\because x_0 \notin E$)

Take $r = \delta - |x - x_0|$.

Since $f(N_r(x_0) \cap E)$ is not bounded, there exists $t \in N_r(x_0) \cap E$ such that $|f(t)| \geq \epsilon + \frac{1}{|x - x_0|}$

Now $|t - x_0| < r$ and, $t \in E$. This implies that $|t - x_0| + |x - x_0| < \delta$ and hence $|x - t| < \delta$

Also $|f(t)| - |f(x)| \geq t$. Thus there exist $x, t \in E$ such that $|x - t| < \delta$ and $|f(t)| - |f(x)| \geq \epsilon$.

Therefore f is not uniformly continuous on E.

So (a) and (c) are proved.

(b) Define $g: E \to \mathbb{R}$ as $g(x) = \frac{1}{1+(x-x_0)^2}$, for all $x \in E$.

Then g is continuous on E. Also $0 < g(x) < I$ for all $x \in E$.

This implies g is bounded.

Now, we will show that $\sup_{x \in E} g(x) = 1$

Clearly, I is an upper bound of $\{g(x) | x \in E\}$.

Now we will show that $p \geq 1$.

If possible suppose that $p < 1$. Then $0 < p < 1$. Now we will show that there exists $x \in E$ such that $g(x) > p$.

Take $\epsilon = \sqrt{\frac{1}{p} - 1}$, since x_0 is a limit point of $E, x \in N_\epsilon(x_0) \cap E \backslash \{x_0\} \neq \emptyset$. Choose $x \in N_\epsilon(x_0) \cap E \backslash \{x_0\}$. Then $x \in E$ and

$$|x - x_0| < \epsilon = \sqrt{\frac{1}{p} - 1} \Rightarrow |x - x_0|^2 < \frac{1}{p} - 1$$

$$\Rightarrow \frac{1}{p} > 1 + |x - x_0|^2 \Rightarrow \frac{1}{1 + (x - x_0)^2} > p$$

$$\Rightarrow g(x) > p$$

Thus there exists $x \in E$ such that *g(x)>p,* which is a contradiction to the fact that p is an upper bound of the set $\{g(x) | x \in E\}$. Therefore $p \geq 1$. Hence $\sup_{x \in E} g(x) - 1$.

This shows that g has no maximum.

Thus if E is bounded, then *(a),(b)* and *(c)* are proved.

Case (ii) : Suppose E is not bounded.

(a) Define $f: E \to \mathbb{R}$ *as* $f(x) - x$ for all $x \in E$. Then f is continuous on E and I is not bounded on E.

So (a) is proved.

as $f(x) = x$ for all $x \subset E$. Then f is continuous on E and I is not

(b) Define $h: E \to \mathbb{R}$ as $h(x) = \frac{x^2}{1+x^2}$ for all $x \in E$.

Then h is continuous on E. Since $h(x) < 1$ for all $x \in E$, h is bounded.

Now we will show that h has no maximum. For this we will show that $\sup_{x\in E} h(x) = 1$.

Since $h(x) < 1$ for all $x \in E$, we have 1 is an upper bound of $\{h(x) \mid x \in E\}$. Let p be any upper bound of $\{h(x) \mid x \in E\}$. If possible suppose that $p < 1$. Then $0 < p < 1$.

Now we will show that there exists $x \in E$ show that $h(x) > p$.

Since E is not bounded, there exists $x \in E$ such that

$$|x| > \sqrt{\frac{p}{1-p}} \Rightarrow x^2 > \frac{p}{1-p} \Rightarrow (1-p)x^2 > p$$

$$\Rightarrow x^2 - px^2 > p \Rightarrow x^2 > p + px^2 = p(1+x^2)$$

$\Rightarrow \frac{x^2}{1+x^2} > p \Rightarrow h(x) > p$, which is a contradiction to the fact that p is an upper bound of $\{h(x) \mid x \in E\}$.

Therefore $p < 1$ and hence $\sup_{x\in E} h(x) = 1$

Thus h is no maximum.

Note: (c) Would be false if boundedness were omitted from the hypothesis.

5.1.17 Example: Let E be the set of all integers. Then E is a non-compact subset of $\mathbb{R}$ which is not bounded. Then every function defined on E is uniformly continuous. For, let f be any function from E into $\mathbb{R}$. Let $\epsilon > 0$. Choose δ such that $0 < \delta < 1$. Suppose $x, y \in E$ such that $|x - y| < \delta$. Then $x - y$. This implies $|f(x) - f(y)| - 0 < \delta$. Hence f is uniformly continuous on E.

5.1.18 Examples :

(1) Let $f(x) = \frac{1}{x}, 0 < x_0 < 1, 0 < x_0 < 1$

Let $\epsilon = \frac{1}{2x_0}$, then there exists a $\delta_1 > 0$

$$|x - x_0| < \delta_1 \Rightarrow ||x| - |x_0|| \le |x - x_0| < \frac{1}{2}|x| > |x_0|$$

Now $\left|\frac{1}{x} - \frac{1}{x_0}\right| = \left|\frac{x - x_0}{x x_0}\right| < \frac{2}{|x^2|}(|x| - |x_0|) < \epsilon$

If $\left|\frac{1}{x} - \frac{1}{x_0}\right| < \frac{\epsilon |x_0|^2}{2}$

If $\delta < \min\left(\delta_1, \frac{\epsilon x_0^2}{2}\right)$, then f is a continuous on $(0, 1)$.

We will show f is not uniformly continuous on $(0, 1)$ given $\epsilon = 1$. Let $\delta > 0$ be as in uniform continuity of f. Choose n such that $\frac{1}{n} < \delta$.

$\frac{1}{n+1} < \frac{1}{n} < \delta$ and $\left|\frac{1}{n+1} - \frac{1}{n}\right| = \left|\frac{1}{n(n+1)}\right| < \frac{1}{n} < \delta$

$\therefore \left|f\left(\frac{1}{n+1}\right) - f\left(\frac{1}{n}\right)\right| = |n+1-n| = 1 \geq \epsilon$

$\Rightarrow f$ is not uniformly continuous on $(0,1)$.

(2) let $f(x) = \sin\left(\frac{1}{x}\right), (0 < x < 1)$

Let $0 < c < 1$, let $\delta > 0$ be as in uniform continuous. Choose '+ve' integer n_1 and n_2 such that

$x_1 = \frac{1}{\left(2n_1+\frac{1}{2}\right)\pi} < \delta$ and $\frac{1}{2n_1\pi} < \delta$

As before $|x_1 - x_2| < \delta$ and $|f(x_1) - f(x_2)| = |1-0| = 1 \geq \epsilon$

$\therefore f$ is not uniformly continuous on $(0,1)$.

(3) let $f(x) = \frac{1}{x}, 0 < a < x < \alpha$

If $|x - x_0| < \epsilon a^2$ and let $\delta = \epsilon a^2$

Then f is uniformly continuous on (a, ∞).

5.2 CONTINUITY AND CONNECTEDNESS

5.2.1 Theorem: If f is a continuous mapping of a metric space X into a metric space Y a if E is a connected subset of X, then $f(E)$ is connected.

Proof: Suppose f *is* a continuous mapping of a metric space X into a metric space Y and is a connected subset of X.

Claim: $f(E)$ is a connected subset of Y

If possible suppose that $f(E)$ is not connected. Then there exist non-empty subsets A and B of Y such that that $f(E) = A \cup B$ and $\bar{A} \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$

$G = E \cap f^{-1}(A)$ and $H = E \cap f^{-1}(B)$

$G \neq \emptyset$ and $H \neq \emptyset$

Since A and B are non-empty, we have. Now consider

$G \cup H = [E \cap f^{-1}(A)] \cup [E \cap f^{-1}(B)]$

$\quad = E \cap f^{-1}(A) \cup f^{-1}(B)$

$\quad = E \cap [f^{-1}(A \cup B)] = E$

$\therefore E = G \cup H$

Now, we will show that $G \subseteq f^{-1}(\bar{A})$

Let $x \in G \Rightarrow x \in E \cap f^{-1}(A) \Rightarrow x \in E$ and $f(x) \in A$

$\rightarrow f(x) \in \bar{A} \rightarrow x \in f(\bar{A})$. Therefore $G \subseteq f^{-1}(\bar{A})$

Since A is a closed set in Y and since f is continuous, by corollary 4.3.13, $f^{-1}(\bar{A})$ is a closed set in X.

Since $f^{-1}(\bar{A})$ is a closed set containing G and $\bar{G}$ is the smallest closed set containing G

we have $\bar{G} \subseteq f^{-1}(\bar{A})$

This implies $f(\bar{G}) \subseteq (\bar{A})$

Next we will show that $f(H) = B$

Let $y \in f(H)$. Then $y = f(x)$ for some $x \in H$.

$x \in H \Rightarrow x \in E$ and $x \in f^{-1}(B) \Rightarrow f(x) \in B \Rightarrow y \in B$.

So $f(H) \subseteq B$.

Let $y \in B \Rightarrow y \in f(E) \Rightarrow Y = f(x)$ for some $x \in E$.

$\Rightarrow x \in f^{-1}(B)$ and $x \in E \Rightarrow x \in E \cap f^{-1}(B)$

$\Rightarrow x \in H \Rightarrow f(x) \in f(H) \Rightarrow y \in f(H)$

So $B \subseteq f(H)$ and hence $f(H) = B$.

Next, we will show that $\bar{G} \cap H = \emptyset$.

If possible suppose that $\bar{G} \cap H = \emptyset$. Then choose and $x \in \bar{G} \cap H$

$\Rightarrow x \in \bar{G}$ and $x \in H \Rightarrow x \in \bar{G}$ and $f(x) \in f(H) = B$.

$\Rightarrow f(x) \in f(\bar{G})$ and $f(x) \in B \Rightarrow f(x) \in A$ and $f(x) \in B$

$(\because f(\bar{G}) \subseteq A)$

$f(x) \in \bar{A} \cap B \neq \emptyset$, a contradiction.

So $\bar{G} \cap H = \emptyset$.

Similarly, we can show that $G \cap \bar{H} = \emptyset$

Therefore $E = G \cup H$ such that $\bar{G} \cap H = \emptyset$ and $G \cap \bar{H} = \emptyset$

Thus E is the union of two separated sets; which is a contradiction to the fact that E is connected. This contradiction arises due to our supposition $f(E)$ is not connected. Hence $f(E)$ is connected.

5.2.2 Theorem: Let f be a real continuous function on the closed interval $[a, b]$. If $f(a) < f(b)$ and if c is a number such that $f(a) < c < f(b)$. then there exists a point $x \in (a, b)$ such that $f(x) = c$.

Proof: Given that f is a continuous real function on the closed interval $[a, b]$.

Suppose, $f(a) < f(b)$ and c is a number such that $f(a) < c < f(b)$.

By a known theorem, $[a, b]$ is connected. Since f is continuous, by Theorem 5.2.1, $f[a, b]$ is connected subset of $\mathbb{R}$. Then by a known theorem, $f[a, b]$ *is* an interval. Since $f(a) < c < f(b)$ and $f(a), f(b) \in f[a, b]$. we have $c \in f[a, b] \Rightarrow c = f(x)$ for some $x \in [a, b]$.

5.2.3 Note: Theorem **5.2.2** holds if $f(a) > f(b)$.

5.2.4 Definition: If f is defined on E, then the set $\{(x, f(x)) | x \in E\}$ is called the graph of f.

5.2.5 Problem: If f is a real valued function defined on a set E of real numbers and if E is compact, then show that f is continuous on E if and only if the graph of f is compact.

Solution: Suppose f is a real valued function defined on a set E of real numbers and also suppose that E is compact.

Claim: f is continuous on E if and only if the graph of f is compact.

Suppose f is continuous on E. Then by Theorem 5.1.2. $f(E)$ is compact. Since the product of a non-empty family of compact sets is compact. We have $EX\ f(E)$ is compact. Since every closed subset of a compact set is compact to show the graph of f is compact, it is enough if we show that the graph of f is a closed subset of $EXf(E)$.

Write $G = \{(x, f(x)) | x \in E\}$. Then G is the graph of f Let $(x, y) \in EXf(E)$ be a limit point of G. Then there exist a sequence $\{(x_n, f(x_n))\}$ of points in G such that $\lim_n (x_n, f(x_n)) = (x, y)$. This implies $\lim_n f(x_n) = x$ and $\lim_n f(x_n) = y$. Since f is continuous and $\lim_n x_n = x$. we have $\lim_n f(x_n) = f(x)$. Since the limit of a sequence is unique, we have $f(x) = y$.

Therefore $(x, y) = (x, f(x)) \in G$. This shows that G contains all of its limit points and hence G is a closed subset of $EXf(E)$. Consequently G is compact. That is, the graph of f *is* compact.

Conversely suppose that the graph G of f *is* compact.

We will show that f is continuous.

Since G is compact, by a known result, G is closed and bounded, Let $x \in E$.

Let $\{x_n\}$ be a sequence of points in E such that $\{x_n\}$ converges to x, Now $\{(x_n, f(x_n))\}$ is a sequence of points in G. Since G is bounded, $\{(x_n, f(x_n))\}$ is bounded.

This implies that $\{f(x_n)\}$ is bounded. Then $\limsup f(x_n)$ and $\liminf f(x_n)$ exist. So let $\infty = \limsup f(x_n)$. Then there exists a sub sequence $\{f(x_{n_k})\}$ of $\{f(x_n)\}$ such that $\{f(x_{n_k})\}$ converges to ∞.

Since $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ and $\{x_n\}$ converges to x, we have $\{x_{n_k}\}$ converges to x Then $\{x, \alpha\} = \lim_k \left(x_{n_k}, f(x_{n_k})\right)$. Now $\lim_k \left(x_{n_k}, f(x_{n_k})\right)$ is a sequence of points in G such that $(x, \alpha) = \lim_k \left(x_{n_k}, f(x_{n_k})\right)$. This implies that (x, α) is a limit point of G. Since G is closed. $(x, \alpha) \in G$ and hence $(x, \alpha) = (x, f(x))$.

Therefore, $\left(x, f(x)\right) = \lim_k sup\{x_{n_k}, f(x_{n_k})\}$

Similarly we can show that $\left(x, f(x)\right) = \lim_k inf\{x_{n_k}, f(x_{n_k})\}$

Therefore, $\left(x, f(x)\right) = \lim_k sup\{x_{n_k}, f(x_{n_k})\}$.

Consequently $\lim_n\ f(x) = f(x)$. So f *is* continuous at x.

Since $x \in E$ is arbitrary, f is continuous on E. Thus f *is* continuous on E if and only if the graph of f is compact.

5.2.6 Problem: Let $1 = [0,1]$ be the closed unit interval. Suppose f is a continuous mapping of I into I. Prove that $f(x) = x$ for at least one $x \in I$.

Solution: Given that $I = [0, 1]$ be the closed unit interval and f is a continuous mapping of I into I.

Define $g: I \to \mathbb{R}$ as $g(x) = x - f(x)$ for all $x \in [0,1]$

Since f is a continuous function, we have g is also a continuous function.

Consider $g(0) - 0 - f(0) \leq 0$ and $g(1) - 1 - f(1) \geq 0$ $(\because 0 \leq f(0))$ and $f(1) \leq 1$.

$\because g(0) \leq 0 < g(1)$.

If $g(0) = 0$, then $0 - f(0) = 0 \Rightarrow f(0) = 0$.

If $g(1) = 0$, then $1 - f(1) = 0 \Rightarrow f(0) = 1$.

Suppose $g(0) < 0 < g(1)$. Then, by known theorem, there exists $x \in (0,1)$ such that $g(x) = 0$. This implies $x - f(x) = 0$ and hence $f(x) = x$.

Thus, in any case, $f(x) = x$ for some $x \in I$.

5.2.7 Problem: Show that a uniformly continuous function of a uniformly continuous function is uniformly continuous.

Solution: Let $\{(X,d_1),(Y,d_2)$ and $(Z,d_3)\}$ be metric spaces. Suppose $f: X \to Y$ and $g: Y \to Z$ are uniformly continuous functions.

Claim: $g \circ f: X \to Z$ is uniformly continuous.

Let $\epsilon > 0$. Since $g: Y \to Z$ is uniformly continuous there exists a $\eta > 0$ such that $d_3(g(y_1), g(y)) < \epsilon$ whenever $d_1(y_1, y_2) < \eta$.........(1)

Since $f: X \to Y$ is uniformly continuous there exists a $\delta > 0$ such that $d_2(g(x_1), g(y)) < \eta$ whenever $d_1(x_1, x_2) < \delta$

Suppose $x_1, x_2 \in X$ such that $d_1(x_1, x_2) < \delta$……….(2)

Then from (1) and (2) $d_3(g \circ f(x_1),\ g \circ f(x_2)) < \epsilon$

Therefore $g \circ f: X \to Z$ is uniformly continuous.

5.2.8 Problem: If E is a non-empty subset of a metric space (X, d) define the distance from $x \in X$ to E by

$$P_E(x) = \inf_{z \in E} d(x,z)$$

5.2.8.1 Prove that $P_E(x) = 0$ if and only if $x \in \bar{E}$

5.2.8.2 Prove that P_E is a uniformly continuous function on x, by showing that $|P_E(x) - P_E(y)| \leq d(x,y)$ for all $x, y \in X$.

Solution: Suppose E is a non-empty subset of a metric space (X, d).

Define $P_E(x) = \inf_{z \in E} d(x, z)$ for all $x \in X$.

(a) To show $P_E(x) = 0$ if and only if $x \in \bar{E}$

Suppose $x \in \bar{E}$

Now $d(x, x) = 0$

If $x \in E$. Then $0 \le P_E(x) \le d(x, x) = 0 \Rightarrow P_E(x) = 0$.

Suppose $x \notin E$. Since $x \in \bar{E}, x$ is a limit point of E. Then there exists a sequence $\{x\}$ of points in E such that $\lim_{n \to \infty} x_n = x$

Let $\epsilon > 0$. Since $\lim_{n \to \infty} x_n = x$. There exists a positive integer N such that $d(x, x) < \epsilon$ for all $n \ge N$.

Now $0 \le P_E(x) \le d(x_n, x)$ for all $n \ge N$.

$\Rightarrow 0 \ge P_E(x) < \epsilon$

Since $\epsilon > 0$ is arbitrary, We have $P_E(x) = 0$.

Thus if $x \in E$ it, then $P(x) = 0$

Conversely suppose that $P_E(x) = 0$.

Let $\epsilon > 0$. Since $P_E(x) = 0$ there exists $y \in E$ such that $d(x, y) < \delta$. This implies $y \in N_E(x)$.

This shows that $N_E(x) \cap E \neq \emptyset$ for any $\epsilon > 0$ and hence $x \in \bar{E}$.

Thus $x \in \bar{E}$ if and only if $P_E(x) = 0$.

(b) To shows P_E is uniformly continuous on X.

Let $\epsilon > 0$. Take $\delta = \epsilon$. Suppose $x, y \in X$ such that $d(x, y) < \delta$.

Consider $P_E(x) \le d(x, z)$ for all $z \in E$.

$\le d(x, y) + d(y, z)$ for all $z \in E$.

$\Rightarrow P_E(x) - d(x, y) \le (y, z)$ for all $z \in E$.

$\Rightarrow P_E(x) \le d(x, y)$ is a lower bound of $\{d(y, z) | z \in E\}$.

$\Rightarrow P_E(x) - d(x, y) \le P_E(y) \Rightarrow P_E(x) - P_E(y) \le d(x, y) < \delta = \epsilon$ Similarly $P_E(y) - P_E(x) < \epsilon$

Therefore $|P_E(x) - P_E(y)| < \epsilon$ whenever $d(x, y) < \delta$.

Hence, P_E is uniformly continuous on X.

Short Answer Questions

1. When do you say that a mapping f of a metric space X into $\mathbb{R}^k$ is bounded?

2. Define a homomorphism of a metric space Into another metric space.

3. When do you say that a function f of a metric space X into a metric space Y is uniformly continuous?

4. Is every uniformly continuous function a continuous function? Justify your answer.

5. Is every continuous function a uniformly continuous function? Justify your answer.

Model Examination Questions

1. If f *is* a continuous mapping of a compact metric space X into a metric space Y then show that $f(X)$ is compact. (Equivalently show that continuous image of a compact metric space is compact).

2. Show that a continuous 1-1 mapping of a compact metric space X onto a metric space Y is a homomorphism.

3. Show that a continuous mapping of a compact metric space X into a metric space Y is uniformly continuous.

4. Let E be a non-compact set in $\mathbb{R}$.Then show that

(i) There exists a continuous function on E which is not bounded.

(ii) There exists a continuous and bounded function on E which has no maximum.

(iii) If, in addition, E is bounded, then show that there exists a continuous function on E which is not uniformly continuous.

5. Show that continuous image of a connected set is connected.

6. Let f be a real continuous function on the closed interval $[a, b)$. If $f(q) < f(b)$ and if c is a number. such that $f(a) < c < f(b)$. then show that there exists a point $x \epsilon (a, b)$ such that $f(X) = c$.

7. If f is a real valued function defined on a set E of real numbers and if E is compact, then show that f is continuous on E if and only if the graph of f is compact.

Exercises

1. Let f be a real uniformly continuous function on the bounded set E in $\mathbb{R}$. Prove that f is bounded on E. Show that the conclusion is false if boundedness of E is omitted from the hypothesis.

2. Suppose f *is* a uniformly continuous mapping of a metric X into a metric space Y. Then prove that $\{f(x_n)\}$ is a Cauchy sequence in Y for every Cauchy sequence $\{x_n\}$ in X.

3. Let E be a dense subset of a metric space X and let f be a, uniformly continuous real function defined on E. Prove that! has a continuous extension from E to X.

4. Call a mapping f of a metric space X into a metric space Y open if $f(V)$ is an open set in Y whenever V is an open set in Prove that every continuous open mapping of is

$\mathbb{R}$ monotonic.

Answers to Short Answer Questions

For **1**, see definition **5.1.1**.

For **2**, see definition **5.1.9**

For **3**, see definition **5.1.1l**

For **4**, see note **5.1.11**

For **5**, see note **5.1.11**

5.3 SUMMARY:

This lesson provides an in-depth examination of continuity, compactness, and connectedness, crucial concepts in topology. Learners will engage with definitions, theorem proofs, and corollaries to solidify their understanding. Practice problems will help learners develop problem-solving skills, enabling them to apply these concepts to mathematical and real-world problems.

5.4 TECHNICAL TERMS:

- Arbitrary
- Argument
- Bounded
- Boundedness
- Closed set
- Compact metric space
- Continuous mapping
- Homomorphism
- Integer
- Limit point
- Lower bound
- Maximum
- Minimum
- Neighbourhood
- Non-compact set
- Open set
- Real number
- Uniform
- Upper bound

5.5 SELF ASSESSMENT QUESTIONS:

1. Define a homomorphism of a metric space Into another metric space.
2. Is every uniformly continuous function a continuous function? Justify your answer.

3. Is every continuous function a uniformly continuous function? Justify your answer.
4. Show that a continuous 1-1 mapping of a compact metric space X onto a metric space Y is a homomorphism.
5. Show that continuous image of a connected set is connected.
6. Suppose f *is* a uniformly continuous mapping of a metric X into a metric space Y. Then prove that $\{f(x_n)\}$ is a Cauchy sequence in Y for every Cauchy sequence $\{x_n\}$ in X.
7. Call a mapping f of a metric space X into a metric space Y open if $f(V)$ is an open set in Y whenever V is an open set in Prove that every continuous open mapping of is $\mathbb{R}$ monotonic.

5.6 SUGGESTED READINGS:

1. Principles of Mathematical Analysis, Third Edition, Mc Graw-HillInternational Editions Walter Rudin.

2. Mathematical Analysis by Tom M. Apostol, Narosa Publishing House, 2nd Edition, 1985.

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LESSON-6

DISCONTINUITIES OF REAL FUNCTIONS

OBJECTIVES:

The objective of the lesson is to help the learners to understand

- ❖ To determine the discontinuity of real functions at a point, and investigate types and properties of discontinuities.
- ❖ To analyze and identify properties of discontinuous functions that develops mathematical reasoning and problem solving skills.

STRUCTUE:

6.0 INTRODUCTION:

Throughout this lesson $f(x)$ denotes a real valued function of real variable. In this lesson the discontinuity of first kind and the discontinuity of second kind are defined. It is proved that if f is a monotonically increasing function defined on (a, b), then $f(x+)$ and $f(x-)$ exist at every point x of (a, b). It is also proved that if f *is* monotonic on (a, b), then the set of points at which f is discontinuous is at most countable.

6.1 DISCONTINUITIES:

6.1.1 Definition: Let f be a function from a metric space X into a metric space Y. If f *is* not Continuous at a point $x \in X$, then we say that f *is* discontinuous at x .

6.1.2 Definition: Let f be a real valued function defined on (a,b). Let x be a point such that $< x \leq b$. A number q is called the right hand limit of f at x iff $f(t_n) \to q$ as $n \to \infty$ for all sequences $\{t_n\}$ in (x, b) such that $t_n \to x$ and we write $f(x+) = q$.

6.1.3 Definition: Let f be a real valued function defined on (a,b). Let x be a point such that $< x \leq b$. A number p is called the left hand limit of f at x

if $f(t_n) \to p$ as $n \to \infty$ for all sequences $\{t_n\}$ in (a, x) such that $t_n \to x$ and we write $f(x-) = p$.

6.1.4 **Note:** If $x \in (a, b)$, then $\lim_{t\to x} f(t)$ exists if and only if

$f(x+) = f(x-) = \lim_{t\to x} f(t)$.

6.1.5 **Definition**: Let f be a real valued function defined on (a,b). If f is discontinuous at a point $x \in (a,b)$ and if $f(x+)$ and $f(x-)$exist, then f is said to have a discontinuity of the first kind or a simple discontinuity at x. In this caseeither $f(x+) \neq f(x-)$ in which case the value of $f(x)$ is immaterial or $f(x+) = f(x-) \neq f(x)$.

6.1.6 **Definition:** Let f be a real valued function defined on (a, b). If f is discontinuous at $x \in (a,b)$ and if either $f(x+)$ or $f(x-)$ does not then f is said to have discontinuity of second kind.

6.2 MONOTONIC FUNCTIONS:

6.2.1 **Definition:** Let f be a real valued function defined on (a,b). Then f is said to be monotonically increasing on (a,b) if $a < x < y < b$ implies that $f(x) \leq f(y)$ *andf is,* said to be monotonically decreasing on (a,b). if $a < x < y < b$ implies that $f(y) \leq f(x)$. f *is* said to be a monotonic function if it is either monotonically increasing or monotonically decreasing.

6.2.2 **Theorem:** Let f be a monotonically increasing function defined on (a,b). Then $f(x+)$ *and* $f(x-)$ exist at every point x of (a,b). More precisely,

$Sup\ f(t) = f(x-) \leq f(x) \leq f(x+) = inf f(t)$

Furthermore, if $a < x < y < b$, then $f(x+) \leq f(y-)$

Proof: Let f be a monotonically increasing function defined on (a,b).

Let $x \in (a,b)$. Since f is monotonically increasing, we have $f(t) \leq f(x)$ for all t such that $a < t < x$. This implies$\{f(t)/a < t < x\}$ is bounded above by $f(x)$. Since $\mathbb{R}$ has least upper bound property, $\{f(t)/a < t < x\}$ has a least upper bound, say A. Then $A \leq f(x)$.

Now we will show that $A - f(x-)$.

Let$\epsilon > 0$. Then $A-\epsilon$is not an upper bound of $\{f(t)/a < t < x\}$. This implies there exists t_0 such that $a < t_0 < x$ and

$A-\epsilon < f(t_0) \leq A$ ……………(1)

Take $\delta = x - t_0$. Then $\delta > 0$. Suppose $t_0 < t < x$. Since f is monotonically increasing, we have

$f(t_0) \leq f(t) \leq A \dots \dots \dots (2)$

From(l) and (2),we have $A - \epsilon < f(t) < A + \epsilon$ whenever $x - \delta < t < x$. This implies $|f(t) - A| < \epsilon$ for all t such that $x - \delta < t < x$ and hence $\lim_{t \to x-} f(t) = A$. Thus $f(x-) = A$. That is, $f(x-) = sup f(t)$.

Next, we will show that $f(x+) = inf\ f(t)$

Since f is monotonically increasing, we have $f(x) \leq f(t)$ for all t such that $x < t < b$. This implies $\{f(t)/x < t < b\}$ is bounded below by $f(x)$. Since $\mathbb{R}$ has greatest lower bound property, $\{f(t)/x < t < b\}$ has a greatest lower bound, say A. Then $f(x) \leq A$.

Now we will show that $A = f(x+)$.

Let $\epsilon > 0$. Then $A + \epsilon$ is not a lower bound of $\{f(t)/x < t < b\}$. This implies there exists t_0 such that $x < t_0 < b$ and

$A \leq f(t_0) \leq A + \epsilon \dots \dots \dots \dots (3)$

Take $\delta = t_0 - x$. Then $\delta > 0$. Suppose $x < t < t_0$. Since f is monotonically increasing, we have

$A \leq f(t) \leq f(t_0) \dots \dots \dots (4)$

From(3) and (4),we have $A - \epsilon < f(t) < A + \epsilon$ whenever $x < t < x + \delta$. This implies $|f(t) - A| < \epsilon$ for all t such that $x < t < x + \delta$ and hence $\lim_{t \to x+} f(t) = A$. Thus $f(x+) = A$. That is, $f(x+) = inf f(t)$.

Hence,

$Sup\ f(t) = f(x-) \leq f(x) \leq f(x+) = inf f(t)$.

Next we will show that $f(x+) \leq f(y-)$ if $a < x < y < b$

Suppose $a < x < y < b$. Then by the above

$f(x+) = inf\ f(t) = inf\ f(t) \dots \dots \dots (5)$

$f(y-) = sup\ f(t) = sup\ f(t) \dots \dots \dots (6)$

From(5) and (6),we have $f(x+) \leq inf\ f(t) \leq sup\ f(t) \leq f(y-)$

Thus if $a < x < y < b$, then $f(x+) \leq f(y-)$.

6.2.3 Note: The above theorem also holds for monotonically decreasing functions.

6.2.4 Corollary: Monotonic functions have no discontinuities of the secondkind.

Proof: Let f be a monotonic function defined on (a, b). Then by theorem 6.2.2 [If f is monotonically increasing) and by note 6.2.3 (if f *is* monotonically decreasing). $f(x+)$ and $f(x-)$ exist at every point $x \in (a, b)$. So f has no discontinuities of second kind.

6.2.5 Theorem: Let f be a monotonic on (a, b).Then the set of points of (a, b) atwhich f is discontinuous is atmost countable.

Proof: Given that f is monotonic on (a, b). Suppose f is monotonically increasing. Let E be the set of points at which f is discontinuous. If E is empty or finite, then E is atmost countable.

Suppose E is not finite. In this case we will show that E is countable.

Let $x \in E$. Then f is discontinuous at x. Since f is monotonic, by corollary 6.2.4, f has discontinuities of first kind. This implies $f(x+), f(x-)$ exist and $f(x-) < f(x+)$ .Then choose a rational number $r(x)$ such that $f(x-) < r(x) < f(x+)$.Thus if $x \in E$, then there exist a rational number $r(x)$ such that $f(x-) < r(x) < f(x+)$.

Write $T = \{r(x) / x \in E\}$. Then $T \subseteq Q$, the set of rational numbers. Since Q is countable, T is also countable ..

Now define $f: E \to T$ *as* $f(x) = r(x)$ for all $x \in E$.

Then clearly f is a function.

Suppose $x_1, x_2 \in E$ such that $x_1 \neq x_2$. Assume $x_1 < x_2$

Then by theorem 4.1.8, $f(x_1+) \leq x_2 -$. This implies that

$$f(x_1-) < r(x_1) < f(x_2+) \leq (x_2-) < r(x_2) < f(x_2+)$$

$\therefore r(x_1) \neq r(x_2)$ and hence $f(x_1) \neq f(x_2)$

Thus $x_1 \neq x_2$ implies that $f(x_1) \neq f(x_2)$

Consequently f *is* one – one.

Clearly f *is* onto

Therefore $f: E \to T$ is a bijection and hence E is countable($\because T$ is countable).

So E is atmost countable.

Now if f *is* a monotonically decreasing function, then $-f$ *is* a monotonically increasing function, then the set of discontinuities of $-f$ are the same, we have the set of discontinuities of f is atmost countable. Thus the set of discontinuities of a monotonic function is atmost countable.

6.3 INFINITE LIMITS AND LIMITS at INFINITY

6.3.1 Definition: For any real c, the set of real numbers x such that $x > c$ is called a neighborhood of $+\infty$ and is written $(c, +\infty)$. For any real c, the set of real numbers x such that $x < c$ is called a neighborhood of $-\infty$ and is written $(-\infty, c)$.

6.3.2 Definition: Let t be a real function defined on E. We say that $f(t) \to A$ as $t \to x$ where A and x are in the extended real number system, if for every neighborhood U of A there is a neighborhood V of x such that $E \cap V$ is non- empty and such that $f(t) \in U$ for all $t \in E \cap V, t \neq x$.

6.3.3 Theorem: $\lim_{t\to x} f(t) = A$ where A and x are extended real numbers if and only if $\lim_{n\to\infty} f(t_n) = A$ for all sequences $\{t_n\}$ in E such that $t_n \neq x$ and $t_n \to x$.

Proof: Suppose $\lim_{t\to x} f(t) = A$

Let $\{t_n\}$ be any sequence in E such that $t_n \neq x$ and $t_n \to x$.

Let U be any neighborhood of A. Since $\lim_{t\to x} f(t) = A$, there exists a neighborhood x such that $V \cap E \neq \emptyset$ and $f(t) \in U$ for all $t \in V \cap E$ and $t \neq x$. Since, $t_n \to x$ there exists a positive integer N such that $t_n \in V$ for all $n \geq N$. This implies $f(t_n) \in U$ for all $n \geq N$ and hence $\lim_{n\to\infty} f(t_n) = A$.

Conversely suppose that $\lim_{n\to\infty} f(t_n) = A$ for all sequences $\{t_n\}$ in E such that $t_n \neq x$ and $t_n \to x$.

If possible suppose that $\lim_{t\to x} f(t) = A$ there exists a neighborhood U of A such that for every neighborhood V of there exists a point $t \in E$ for which $f(t) \notin U$ and $t \in V$.

Case (i): Suppose $= +\infty$. Let n be a positive integer. Now (n, ∞) is aneighborhood of ∞. Then their exists $t_n \in E$ such that $f(t_n) \notin U$ and $t_n \in (n, \infty)$. Therefore $\{t_n\}$ is a sequence of points in E such that $t_n \to \infty$, $t_n \neq \infty$ and

$\lim_{n\to\infty} f(t_n) \neq A$.

Case(ii): Suppose $x = -\infty$. Let n be a positive integer. Now $(-\infty, -n,)$ is a neighborhood of $-\infty$. Then their exists $t_n \in E$ such that $f(t_n) \notin U$ and $t_n \in (-\infty, -n)$. Therefore $\{t_n\}$ is a sequence of points in E such that $t_n \to \infty$, $t_n \neq -\infty$ and

$\lim_{n\to\infty} f(t_n) \neq A$.

Case (iii): Suppose x is a real number. Let n be any positive integer.

$\left(x-\frac{1}{n}, x+\frac{1}{n}\right)$ is a neighborhood of x. Then their exists $t_n \in E$ such that $f(t_n) \notin U$ and $t_n \in \left(x-\frac{1}{n}, x+\frac{1}{n}\right)$. Therefore $\{t\}$ is a sequence of points in E such that $t_n \to x$ as $n \to \infty$, $t_n \neq x$ and $\lim_{n\to\infty} f(t_n) \neq A$.

Thus in any case there exists a sequence $\{t\}$ of points in E such that $t_n \neq x$ and $\lim_{n\to\infty} f(t_n) \neq A$, which is a contradiction to our supposition. This contradiction arises due to our assumption $\lim_{t\to x} f(t) \neq A$. Hence $\lim_{t\to x} f(t) = A$.

6.3.4 Problem: Define $f:(0,2) \to \mathbb{R}$ as $f(x) = 1$ if $0 < x \leq 1$ and $f(x) = 2$ if $1 < x < 2$. Then show that f is continuous at every point $x \neq 1$ and f has a discontinuity of first kind at $x = 1$.

Solution: First we show that f is continuous at every $x \in (0,2)$ such that $x \neq 1$.

Let $x \in (0,2)$ such that $x \neq 1$ and let $\in > 0$.

Then $0 < x < 1$ or $1 < x < 2$.

Suppose $0 < x < 1$. Choose such that $0 < \delta < min\{x, 1-x\}$.

Then $0 < x-\delta < x < x+\delta < 1$

Suppose $y \in (0,2)$ such that $|x-y| < \delta$. Then $x-\delta < Y < x+\delta$

This implies $0 < y < 1$.

consider $|f(x)-f(y)| = |1-1| = 0 < \epsilon$.

So, in this case, f is continuous at .

Suppose $1 < x < 2$. Choose δ such that $0 < \delta < min\{x-1, 2-x\}$

Then $1 < x-\delta < x < x+\delta < 2$.

Suppose $y \in (0,2)$ such that $|x-y| < \delta$. Then $x-\delta < y < x+\delta$

This implies $1 < y < 2$

Consider $|f(x)-f(y)| = |2-2| = 0 < \epsilon$.

So, in this case also f *is* continuous at .

Thus f is continuous at every point $x \in (0,2)$ such that $x \neq 1$. Next we will showthat f is discontinuous at $x = 1$.

Let $\{t_n\}$ be any sequence in $(1,2)$ such that $t_n \to 1$.

Then $f(1+) = \lim_{n\to\infty} f(t_n) = \lim_{n\to\infty} 2 = 2$. So $f(1+) = 2$

Let $\{t_n\}$ be any sequence in $(0,1)$ such that $t_n \to 1$.

Then $f(1-) = \lim_{n\to\omega} f(t_n) = \lim_{n\to\omega} 1 = 1$. So $f(1-) = 1$.

Therefore $f(1+)$ and $f(1-)$ exist and $f(1+) \neq f(1-)$.

So f has a discontinuity of first kind at $x = 1$.

6.3.5 Problem: Define $f: \mathbb{R} \to \mathbb{R}$ as $f(x) = 1$ if x is a rational number and $f(x) = 0$ if x is a irrational numbers. Then show that f has a discontinuity of second kindat every point $x \in \mathbb{R}$.

Solution: First we show that f is discontinuous at every point $x \in \mathbb{R}$.

Let $x \in \mathbb{R}$ and let $0 < \epsilon < 1$.

Let δ be any real number such that $\delta > 0$.

Case (i) : Suppose x is a rational number.

Choose an irrational number y such that $-\delta < y < x + \delta$.

Then $|x - y| < \delta$.

Consider $|f(x) - f(y)| = |1 - 0| = 1 > \epsilon$

Case (ii): Suppose x is an irrational number.

Choose a rational number y such that $-\delta < y < x + \delta$.

Then $|x - y| < \delta$.

Consider $|f(x) - f(y)| = |1 - 0| = 1 > \epsilon$

Thus in any case, for $0 < \epsilon < 1$, for any $\delta > 0$, there exists $y \in (x - \delta, x + \delta)$ such that $|f(x) - f(y)| > \epsilon$.

This shows that f is discontinuous at x.

Hence f is discontinuous at every point $x \in \mathbb{R}$

Next, we will show that f has a discontinuity of second kind at every point $x \in \mathbb{R}$.

Let $x \in \mathbb{R}$. For each positive integer n, consider $\left(x, x + \frac{1}{n}\right)$

Choose a rational number r_n in $\left(x, x + \frac{1}{n}\right)$. Then $\{r_n\}$ is a sequence of rational numbers such that $r_n \to x$.

$Since$, $\lim_{n\to\infty} \left(x + \frac{1}{n}\right) = x$ and $x < r_n < \frac{1}{n}$.

Consider $\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} 1 = 1$.

So $\{r_n\}$ is a sequence of rational numbers in (x, ∞) such that $r_n \to x$ and

$\lim_{n\to\infty} f(r_n) = 1$. Let $\{s_n\}$ be a sequence of irrational numbers such that $x < s_n < x + \frac{1}{n}$.

Then $\{s_n\}$ be a sequence of irrational numbers in (x, ∞) such that $s_n \to x$ as $n \to \infty$ and

$\lim_{n\to\infty} f(s_n) = \lim_{n\to\infty} 0 = 0$.

Thus $\{r_n\}$ and $\{s_n\}$ are two different sequences in (x, ∞) such that $r_n \to x$ and $s_n \to x$ but $\lim_{n\to\infty} f(s_n) = 0 \neq 1 = \lim_{n\to\infty} f(r_n)$.

This shows that $f(x+)$ does not exist and f has a discontinuity of second kind at x. Hence f has a discontinuity of second kind at every point.

6.3.6 Problem: Define $f:\mathbb{R} \to \mathbb{R}$ *as* $f(x) = 0$ if x is irrational and $f(x) = 0$ if x is irrational. Then show that f is continuous at $x = 0$ and has a discontinuity of the second kind at every other point in $\mathbb{R}$.

Solution: First we show that f is continuous at $x = 0$

Let $\epsilon > 0$. Take $\delta = \epsilon$

Suppose $y \in \mathbb{R}$ such that $|y - 0| < \delta \Longrightarrow |y| < \epsilon$

Consider $|f(y) - f(0)| = |f(y) - 0| = |f(y)| = |y|$ or 0 according *as* y is rational or y is irrational. This implies that $|f(y) - f(0)| < \epsilon$.

Therefore f is continuous at $x = 0$.

Suppose $x \in \mathbb{R}$ such that $x \neq 0$.

For each positive integer n consider $\left(x, x + \frac{1}{n}\right)$

Choose a rational number r_n in $\left(x, x + \frac{1}{n}\right)$. Then $\{r_n\}$ is a sequence of rational numbers such that $r_n \to x$.

Consider $\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} r_n = x$.

For each positive integer n

Choose an irrational number s_n in $\left(x, x + \frac{1}{n}\right)$. Then $\{s_n\}$ is a sequence of irrational numbers such that $s_n \to x$.

and $\lim_{n\to\infty} f(s_n) = \lim_{n\to\infty} 0 = 0$.

Thus $\{r_n\}$ and $\{s_n\}$ are two different sequences in (x, ∞) such that $r_n \to x$ and $s_n \to x$ but $\lim_{n\to\infty} f(s_n) = 0 \neq x = \lim_{n\to\infty} f(r_n)$.

This shows that $f(x+)$ does not exist and f has a discontinuity of second kind at x. Hence f has a discontinuity of second kind at every point $x \neq 0$.

6.4 SOME MORE EXAMPLES WITH SOLUTIONS:

6.4.1 Example: Call a mapping from X into Y open if $f(V)$ is an open set in Y whenever V is an open set in X. Prove that every continuous open mapping of R' into R' is monotonic.

Solution: Suppose f is continuous and not monotonic, say there exist points $a < b < c$ with $f(a) < f(b)$ and $f(c) < f(b)$.

Then the maximum value of f on the closed interval $[a, c]$ is assumed at appoint u in the open interval (a, c).

If there is also a point v in the open interval (a, c) where f assumes its minimum value on $[a, c]$, then $f(a, c) - [f(v), f(u)]$.

If no such point v exists, then $f(a, c) = (d, f(u))$, where $d = \min(f(a), f(c))$

In either case, the image of (a, c) is not open.

6.4.2 Example: Let $[x]$ denote the largest integer contained in x, that is $[x]$ is the integer such that $x - 1 < [x] \leq x$; and let $(x) = x - [x]$ denote the fractional part of x. What discontinuities do the functions $[x]$ and (x) have?

Solution: The two functions have the same discontinuities,

Since each can be written as the difference of the continuous function $f(x) = x$ and the other function.

Now the function $[x]$ is constant on each open interval $(k, k + 1)$;

Hence its only possible discontinuities are the integers.

These are of course real discontinuities, since if $\epsilon = 1$, there is no $\delta > 0$ such that $|[x] - [k]| < \epsilon$ whenever $|x - k| < \delta$.

6.4.3 Example: Let f be a real function defined on (a, b). Prove that the set of points at which f has a simple discontinuity is at most countable. Hint : Let E be the set on which $f(x-) < f(x+)$. With each point x of E associate a triple (p, q, r) of rational numbers such that

(a) $f(x-) < p < f(x+)$,

(b) $a < q < t < x$ implies $f(t) < p$,

(c) $x < t < r < b$ implies $f(t) > p$.

The set of such triples is countable. Show that each triple is associated with at most one point of E. Deal similarly with the other possible types of simple discontinuities.

Solution: The existence of three such rational numbers (p, q, r) for each simple discontinuity of this type follows from the n assumption $f(x-) < f(x+)$, and the definition of $f(x-)$ and $f(x+)$.

We need to show that a given triple (p, q, r) cannot be associated with any other discontinuity of this type.

To that end suppose $y > x$ and $f(y-) < f(y+)$.

If we do not have $f(y-) < p < f(y+)$, then the triple chosen for y will differ from (p, q, r) in the first element.

Hence suppose $f(y-) < p < f(y+)$.

In this case we definitely cannot have $r > y$, since there are points $t \in (x, y)$ such that $f(t) < p$ (if there weren't, we would have $f(y-) \geq p$.

We have thus shown that shown that the set of points $x \in (a, b)$ at which $f(x-) < f(x+)$ is at most countable.

The proof that the set of points at which $f(x-) > f(x+)$ is at most countable is, of course, nearly identical.

Now consider the set of points x at which $\lim_{t \to x} f(t)$ exists, but is not equal to $f(x)$.

For each point $x \in (a, b)$ such that $\lim_{t \to x} f(t) < f(x)$, we take a triple (p, q, r) of rational numbers such that

(a) $\lim_{t \to x} f(t) < p < f(x)$

(b) $a < q < t < x$ or $x < t < r < b$ implies $f(t) < p$.

As before, if $y > x$ and $\lim_{t \to y} f(t) < f(y)$, the triple associated with y will be different from that associated with x.

For even if $\lim_{t \to y} f(t) < p < f(y)$, we cannot have $r > y$, since $f(y) > p$ and $x < y$.

The proof that the set of points $x \in (a, b)$ at which $\lim_{t \to x} f(t) > f(x)$ is countable is nearly identical.

Hence, the number of points in $[a, b]$ at which f has a discontinuity of first kind is countable.

Short Answer Questions

1. When do you say that a real valued function f defined on $(a. b)$ has a discontinuity of first kind?

2. When do you say that areal valued function f defined on (a, b) has a discontinuity of second kind?

3. Determine $f: \mathbb{R} \to \mathbb{R}$ *as* $f(x) = x$ if r is rational and $f(x) = 0$. If x is irrational. Then show that f is continuous *at* $x = 0$.

Model Examination Questions

1. Let f be a monotonically increasing function defined on (a, b). Then showthat $f(x+)$ and $f(x-)$ exist at every point x of (a, b). More precisely,

2. Let f be monotonic on (a, b). Then show that $\sup f(t) = f(x-) \leq f(x) \leq f(x-) = \inf f(t)$ the set of points of (a, b) at which f is discontinuous is atmost countable.

3. Define $f: (0,2) \to \mathbb{R}$ as $f(x) = 1$ if $0 < x < 1$ and $f(x) = 2$ if $1 < x < 2$ Then show that f is continuous at every point $x \neq 1$ and f has a discontinuity of first kind at $x = 1$

4. Define $f: \mathbb{R} \to \mathbb{R}$ *as* $f(x) = 1$ if x is rational and $f(x) = 0$ if x is irrational. Then show that f has a discontinuity of second kind at every point $x \in \mathbb{R}$

Exercises

1. Suppose X, Y and Z are metric spaces and Y is compact. Let f map X into$Y;$ let g be a continuous one-to-one mapping of Y into Z, and put $h(x) = g\,(f(x))$for all $x \in X$ Prove that f is uniformly continuous if h is uniformly continuous.

Answers to Short Answer Questions

1. For 1, see definition 6.1.5
2. For 2, see definition 6.1.6
3. For 3, see definition 6.2.1

6.5 SUMMARY:

This lesson focuses on understanding and analyzing discontinuities of real functions at a point. Learners will explore types and properties of discontinuities, developing mathematical reasoning and problem-solving skills. The Lesson Highlights Introduction to discontinuities of real functions, Definitions and proofs of relevant theorems, Solved problems and examples to illustrate key concepts, Analysis of properties of discontinuous functions.

6.6 TECHNICAL TERMS:

- Atmost countable
- Bijection

- Discontinuity
- Monotonically decreasing
- Monotonically increasing
- Neighborhood
- Real variable

6.7 SELF ASSESSMENT QUESTIONS:

1. Let f be monotonic on (a,b). Then show that $\sup f(t) = f(x^-) \leq f(x) \leq f(x^-) = \inf f(t)$ the set of points of (a,b) at which f is discontinuous is atmost countable
2. When do you say that a real valued function f defined on $(a.b)$ has a discontinuity of first kind ?
3. When do you say that a real valued function f defined on (a,b) has a discontinuity of second kind ?
4. Let $[x]$ denote the largest integer contained in x, that is $[x]$ is the integer such that $x-1<[x]\leq x$; and let $(x) = x-[x]$ denote the fractional part of x. What discontinuities do the functions $[x]$ and (x) have ?

6.8 SUGGESTED READINGS:

1. Principles of Mathematical Analysis, Third Edition, Mc Graw-Hill International Editions Walter Rudin.

2. Mathematical Analysis by Tom M. Apostol, Narosa Publishing House, 2nd Edition, 1985.

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LESSON -7

DERIVATIVE OF REAL FUNCTIONS

OBJECTIVES:

The objective of the lesson is to help the learners to understand

- ❖ To define and explain derivatives of real functions and apply derivatives to solve optimizations skills.
- ❖ To develop problem solving skills using derivatives.

STRUTURE:

7.1 : INTRODUCTION:

1. **Derivative at a point:** Let I denotes the open interval]a,b[in $\mathbb{R}$ and let $x_0 \in I$. Then a function $f: I \to \mathbb{R}$ is said to be differentiable (or) derivable at x_0 iff

$$\lim_{t \to 0} \frac{f(x_0 + t) - f(x_0)}{t}$$

Or equivalently

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists and is denoted by $f'(x_0)$ or by $Df(x_0)$.

2. Progressive and regressive derivatives:

Definition : The progressive derivatives of f at $x = x_0$ is given by

$$\lim_{t \to 0} \frac{f(x_0 + t) - f(x_0)}{t}, t > 0$$

and is denoted by $Rf'(x_0)$ or by $f'(x_0 + 0)$.

The regressive derivative of f at $x = x_0$ is given by

$$\lim_{t \to 0} \frac{f(x_0 - t) - f(x_0)}{-t}, t > 0$$

and is denoted by $Lf'(x_0)$ or by $f'(x_0 - 0)$.

Progressive and regressive derivatives are also called right hand and left hand differential coefficients of f at $x = x_0$.

It is easy to see that $f'(x_0)$ exists iff $Rf'(x_0)$ and $Lf'(x_0)$ exists and are equal.

3. Differentiability in $[a,b]$. A function $f:[\mathrm{a},\mathrm{b}] \to \mathbb{R}$ is said to be differentiable at a iff $Rf'(a)$ exists, differentiable at b iff $Lf'(b)$ exists. f is said to be differentiable in $[\mathrm{a},\mathrm{b}]$ iff it is differentiable at every point of $[\mathrm{a},\mathrm{b}]$.

4. Derivative of function. Let I^* denoted the subset of I consisting of all points of I at which f is differentiable. Then the function $F: I^* \to \mathbb{R}$ defined by $F(x) = f'(x)$ for all $x \in I^*$ is called the first derivative of f (or simply the derivative of f) and is denoted by f' or by Df. Similarly 2nd, 3rd, …, nth derivatives of f are defined and denoted by $f'', f''', \dots, f^{(n)}$ respectively.

Note: The definitions given in 1 and 4 above concern two different but related concepts. The derivative of f at a point a is a number while the derivative of f is a function. However, very often the term derivative of f is employed to denote both number and function and it is left to context to distinguish which is intended.

7.1.1 Definition: Let f be defined and a real valued function on $\{a,b\}$. For any $x \in [a,b]$ from the quotient.

$$\psi - \frac{f(t)-f(x)}{t-x}, (a < t < b, t \neq x) \text{ ....}(1)$$

and define

$$f'(x) = \lim_{t\to x} \phi(t) \text{.............................}(2)$$

provided this limit exists.

We thus associate with the function f, a function f' whose domain is the set of points x at which limit *(2)* exists; f' is called derivatives of f.

If f' is defined at a point x, we say that f is differentiable at x. If f' defined at every point of set $E \subset [a,b]$, we say that f is differentiable on E.

It is possible to consider right hand and left hand limits is *(2)*; this leads to right hand and left hand derivatives, we shall not, however, discuss one-sided derivatives in any detail.

7.1.2 : Theorem: Let f be defined on $[a,b]$. If f is differentiable at a point $x \in [a,b]$, then f is continuous at x.

Proof: Suppose that 'f' is differentiable at a $x \in [a,b]$.

point we show that f is continuous at x.

Let $t \in [a,b] \ni t \neq x$.

Now $f(t) - f(x) = \frac{f(t)-f(x)}{t-x} x (t-x)$

Taking limit $t \to x$ on both sides

$$\lim_{t\to x}[f(t) - f(x)] = \lim_{t\to x}\left(\frac{f(t)-f(x)}{t-x}\right) x \lim_{t\to x}(t-x)$$

$$= f'(x).0$$

$$= 0$$

$$\Rightarrow \lim_{t \to x} f(t) = f(x)$$

$\therefore f$ is continuous at x.

7.1.3 Remark: The converse of the above theorem is not true is a continuous function need not be differentiable.

Justification: Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = |x|\ \forall x \in \mathbb{R}$

Now we prove that f is continuous at $x = 0$ and f is not differentiable at $x = 0$

Now for any $x \in \mathbb{R}$,

$$f(x) = |x| = \begin{cases} x & if\ x \geq 0 \\ -x & if\ x < 0 \end{cases}$$

R. H. L $= \lim_{x \to 0+} f(x) = \lim_{x \to 0+0} |x|$

$= \lim_{h \to 0} |0 + h|$ where $x = 0 + h$

$= 0$

L. H. L $= \lim_{x \to 0-} f(x) = \lim_{x \to 0-0} |x|$

$= \lim_{h \to 0} |0 - h|$ where $x = 0 - h$

$= 0$

So R. H. L= L. H. L

Hence $\lim_{x \to 0} f(x)$ exists and is 0.

Also, $f(0) = |0| = 0$

Thus $\lim_{x \to 0} f(x) = f(0)$

This shows that f is continuous at $x = 0$

Differentiability:

We have $f(0) = |0| = 0$

Now R. H. D $f'(0+0) = \lim_{t \to 0+0} \frac{f(t) - f(0)}{t - 0}$

$= \lim_{t \to 0+0} \frac{f(t) - 0}{t - 0}$

$= \lim_{t \to 0+0} \frac{|t|}{t}$

$= \lim_{h \to 0} \frac{|0+h|}{0+h}$ where $t = 0 + h$

$= \lim_{h \to 0} \frac{h}{h}$

$= \lim_{h \to 0} (1)$

$= 1$

Now L. H. D $f'(0-0) = \lim_{t \to 0-0} \frac{f(t) - f(0)}{t - 0}$

$= \lim_{t \to 0-0} \frac{f(t) - 0}{t - 0}$

$= \lim_{t \to 0-0} \frac{|t|}{t}$

$= \lim_{h \to 0} \frac{|0-h|}{0-h}$ where $t = 0 - h$

$= \lim_{h \to 0} \frac{h}{-h}$

$= \lim_{h \to 0} (-1)$

$= -1$

So R. H. D$\neq$ L. H. D

$\therefore \lim_{t \to 0} \frac{f(t)-f(0)}{t-0}$ does not exist

That is $f'(0)$ does not exist

This shows that f is not differentiable at 0.

7.1.4 Example:

Take the function $f(x) = |x|$ $[1,-1]$.

clearly f is continuous at $x = 0$.

For $t \neq 0$, $\lim_{t \neq 0} \frac{f(t)-f(0)}{t-0} = \lim_{t \neq 0} \frac{f|t|-0}{t-0} = \lim_{t \neq 0} \frac{f|t|}{t}$,

But $\frac{|t|}{t} \to 1$ as $t \to 0^+$ and $\frac{|t|}{t} \to -1$ as $t \to 0^-$

$\lim_{t \neq 0} \frac{|t|}{t}$ does not exists.

Hence f is not differential at $x = 0$.

7.1.5 Theorem: Suppose that f and g are defined on $[a, b]$ and are differential at a point $x \in [a.b]$. Then $f + g$, fg and $\frac{f}{g}$ are differentiable at x, and

(a) $(f + g)'(x) = f'(x) + g'(x)$

(b) $(fg)'(x) = f'(x)g(x) + g(x)f(x)$

(c) $\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x)-f(x)g'(x)}{g''(x)}$

In (c) we assume that $g(x) \neq 0$.

Proof: Suppose that f and g are differentiable at a point $x \in [a, b]$

(a) let $h = f + g$

we show that h is differentiable at x and $h'(x) = f'(x) + g'(x)$ take some $t \in [a,b] \ni t \neq x$. Then

$$\lim_{t \to x} \frac{h(t) - h(x)}{t - x} = \lim_{t \to x} \left[\frac{(f + g)(t) - (f + g)(x)}{t - x}\right]$$

$$= \lim_{t\to x}\left[\frac{(f)(t)+(g)(t)-(f)(x)-(g)(x)}{t-x}\right]$$

$$= \lim_{t\to x}\left[\frac{(f)(t)-(f)(x)}{t-x}\right]+\lim_{t\to x}\left[\frac{(g)(t)-(g)(x)}{t-x}\right]$$

$$= f'(x)+g'(x)$$

$\therefore h$ is differentiable at x and $h'(x) = f'(x) + g'(x)$. So, $(f+g)'(x) = f'(x)+g'(x)$

(b) let $h = fg$

Consider $t \in [a,b] \ni t \neq x$. Then

$$\lim_{t\to x}\frac{h(t)-h(x)}{t-x} = \lim_{t\to x}\left[\frac{(fg)(t)-(fg)(x)}{t-x}\right]$$

$$= \lim_{t\to x}\left[\frac{f(t)g(t)-f(x)g(x)}{t-x}\right]$$

$$= \lim_{t\to x}\left[\frac{f(t)g(t)-f(t)g(x)+f(t)g(x)-f(x)g(x)}{t-x}\right]$$

$$= \lim_{t\to x}\left[\frac{f(t)\big(g(t)-g(x)\big)+\big(f(t)-f(x)\big)g(x)}{t-x}\right]$$

$$= \lim_{t\to x}\left[\frac{f(t)\big(g(t)-g(x)\big)}{t-x}\right]+\lim_{t\to x}\left[\frac{\big(f(t)-f(x)\big)g(x)}{t-x}\right]$$

$$= f(x)g'(x)+f'(x)g(x)$$

$\therefore h$ is differentiable at x and $h'(x) = f(x)g'(x)+f'(x)g(x)$.

So, $(fg)'(x) = f(x)g'(x)+f'(x)g(x)$.

(c) let $h = \frac{f}{g}$

Take some $t \in [a,b] \ni t \neq x$. Then

$$\lim_{t\to x}\frac{h(t)-h(x)}{t-x} = \lim_{t\to x}\left[\frac{\left(\frac{f}{g}\right)(t)-\left(\frac{f}{g}\right)(x)}{t-x}\right]$$

$$= \lim_{t\to x}\left[\frac{\frac{f(t)}{g(t)}-\frac{f(x)}{g(x)}}{t-x}\right]$$

$$= \lim_{t \to x} \left[\frac{\frac{f(t)g(x) - f(x)g(t)}{g(t)g(x)}}{t - x} \right]$$

$$= \lim_{t \to x} \left[\frac{f(t)g(x) - f(t)g(x) + f(t)g(t) - f(x)g(t)}{g(t)g(x)(t - x)} \right]$$

$$= \lim_{t \to x} \frac{-f(t)\left(-g(x) + g(x)\right) + \left(f(t) - f(x)\right)g(t)}{g(t)g(x)(t - x)}$$

$$= \lim_{t \to x} \frac{-f(t)}{g(t)g(x)} g'(x) + \lim_{t \to x} \frac{g(t)}{g(t)g(x)} f'(x)$$

$$= \frac{-f(x)}{g''(x)} g'(x) + \frac{g(x)}{g''(x)} f'(x)$$

$$= \frac{f'(x)g(x) + f(x)g'(x)}{g''(x)}$$

$\therefore h$ is differentiable $\left(\frac{f}{g}\right)'(x) = h'(x)\frac{f'(x)g(x) - f(x)g'(x)}{g''(x)}$.

7.1.6 Example: The derivative of any constant is zero. If f is defined by $f(x) = x$, then $f'(x) = x$. Repeated application of (b) &(c) this shows that x^n is differentiable and its derivative is nx^{n-1} for any integer n. Thus every polynomial is differentiable and so every rational function is also differentiable.

7.2 : THE CHAIN RULE:

7.2.1 Theorem (Chain rule):

Suppose f is continuous on $[a, b]$, $f'(x)$ exists at same point $x \in [a, b]$, g is defined on an interval I, which $h'(x) = g'\left(f(x)f'(x)\right)$.

Proof: Let $h(t) = g(f(t))$, where $a \leq t \leq b$

Suppose that $f'(x)$ exists at same point $x \in [a, b]$ and g is differentiable at $f(x)$

show that h is differentiable at x and $h'(x) = g'\left(f(x)f'(x)\right)$

Let $y = f(x)$

Define the function U and V by

$U(t) = \frac{f(t) - f(x)}{t - x} - f'(x)$ and

$V(s) = \frac{g(s) - g(y)}{s - y} - g'(x)$

Then

$$\lim_{t\to x} U(t) = \lim_{t\to x}\left[\frac{f(t)-f(x)}{t-x} - f'(x)\right] = 0$$

and

$$\lim_{s\to y} V(s) = \lim_{s\to y}\left[\frac{g(s)-g(y)}{s-y} - g'(x)\right] = 0$$

Let $s = f(t)$

Consider $h(t) - h(x) = g(f(t)) - g(f(x))$

$$= g(s) - g(y)$$

$$= (s-y)[V(s) + g'(y)] \text{ (from (2))}$$

$$= [f(t)-f(x)][v(s) + g'(y)]$$

$$= (t-x)\,(v(t) + f'(x))(v(x) + g'(y)) \qquad \text{(from (1))}$$

$$\Rightarrow \frac{h(t)-h(x)}{t-x} = [U(t) + f'(x)][V(s) + g'(y)]$$

$$\Rightarrow \lim_{t\to x}\frac{h(t)-h(x)}{t-x} = \lim_{t\to x}[U(t) + f'(x)]\lim_{t\to x}[V(s) + g'(y)]$$

$$= f'(x)g'(y).$$

So, h is differentiable at x and $h'(x) = f'(x)g'(y)$

$\therefore h'(x) = g'(f(x)f'(x))$.

7.2.2 Examples:

(a) Let f be defined by

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, \text{ if } (x \neq 0) \\ 0 \quad \text{if} \quad n = 0 \end{cases}$$

Clearly for, f is differentiable at all points $x \neq 0$ and $f'(x) = \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$ but f is not differentiable at $x = 0$.

for $t \neq 0$, consider $\lim_{t\to 0}\frac{f(t)-f(0)}{t-0} = \lim_{t\to 0}\frac{t\sin\left(\frac{1}{t}\right)}{t-0} = \lim_{t\to 0}\sin\left(\frac{1}{t}\right)$

Their limit does not exists

$\therefore$ f is not differentiable at $x = 0$.

(b) Let f be defined by $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, \text{ if } (x \neq 0) \\ 0 \quad \text{if} \quad n = 0 \end{cases}$

clearly f is differentiable at all points $x \neq 0$ and $f'(x) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$

Here f is not differentiable at all points $x = 0$.

for $t \neq 0$, consider $\lim_{t\to 0} \frac{f(t)-f(0)}{t-0} = \lim_{t\to 0} \frac{t^2 \sin\left(\frac{1}{t}\right)}{t-0} = \lim_{t\to 0} \sin\left(\frac{1}{t}\right) = 0$

f is differentiable at $x = 0$ & $f'(0) = 0$.

But the deliverable f' is not continuous since $\lim_{t\to 0} f'(x) = |f| = 0$

For $\lim_{t\to 0} \frac{f(t)-f(0)}{t-0} = 0$

$t \neq 0 \lim_{t\to 0} \frac{t^2 \sin\left(\frac{1}{t}\right)}{t-0} = 0$

$= \lim_{t\to 0} \sin\left(\frac{1}{t}\right)$

f is differentiable at $x = 0 \perp f'(0) = 0$, but the differentiable f'

(c) Let $x_0 \in R^+ +$ and $f(x) = |x - x_0|$. The function f is continuous at each $x_0 \in R^+$, but $f'(x_0)$ does not exists, since $f'_+(x_0) = 1$ and $f'_-(x_0) = 1$, for $x_0, x_1 \in R^+$, $x_0 \neq x_1$.

7.3 SOME MORE EXAMPLES WITH SOLUTIONS:

7.3.1 Example: Let f be defined for all real x, and suppose that
$|f(x) - f(y)| \leq (x-y)^2$

for all real x and y. Prove that f is constant.

Solution: Dividing by $x - y$, and letting $x \to y$, we find that $f'(y) = 0$ for all y.

Hence f is constant.

7.3.2 Example: Suppose $f'(x) > 0$ in (a,b). Prove that f is strictly increasing in (a,b), and let g be its inverse function. Prove that g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)} \quad (a < x < b).$$

Solution: For any c, d with $a < c < d < b$ there exists a point $p \in (c,d)$ such that $f(d) - f(c) = f'(p)(d-c) > 0$.

Hence $f(c) < f(d)$.

We know that the inverse function g is continuous. (Its restriction to each closed subinterval $[c,d]$ is a continuous, and that is sufficient.)

Now observe that if $f(x) = y$ and $f(x+h) = y+k$, we have

$$\frac{g(y+k)-g(y)}{k} - \frac{1}{f'(x)} = \frac{1}{\frac{f(y+h)-f(x)}{h}} - \frac{1}{f'(x)}.$$

Since we know that

$$\lim \frac{1}{\varphi(t)} = \frac{1}{\lim \varphi(t)}$$

provided $\lim \varphi(t) \neq 0$, it follows that for any $\in > 0$ there exists $\eta > 0$ such that

$$\left| \frac{1}{\frac{f(y+h)-f(x)}{h}} - \frac{1}{f'(x)} \right| < \epsilon$$

if $0 < |h| < \eta$. Since $h = g(y+k) - g(y)$, there exists $\delta > 0$ such that $0 < |h| < \eta$ if $0 < |h| < \delta$.

The proof is now complete.

7.3.3 Example: If

$$C_0 + \frac{C_1}{2} + \cdots\cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} - 0,$$

where $C_0, \cdots\cdots\cdots\cdots, C_n$ are real constants, prove that the equation

$$C_0 + C_1 x + \cdots\cdots + C_{n-1}x^{n-1} + C_n x^n = 0$$

Has at least one real root between 0 and 1.

Solution: Consider that polynomial

$$p(x) = C_0 x + \frac{C_1}{2}x^2 + \cdots\cdots + \frac{C_{n-1}}{n}x^n + \frac{C_n}{n+1}x^{n+1},$$

It is obvious that $p(0) = 0$, and the hypothesis of the problem is that $p(1) = 0$.

Hence Rolle's theorem implies that $p'(x) = 0$ for some x between 0 and 1.

7.3.4 Example: Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.

Solution: Let $\epsilon > 0$. Choose x_0 such that $|f'(x)| < \epsilon$ if $x > x_0$.
Then for any $x \geq x_0$ there exists $x_1 \in (x, x+1)$ such that

$$f(x+1) - f(x) = f'(x_1).$$

Since $|f'(x_1)| < \epsilon$, it follows that $|f(x+1) - f(x)| < \epsilon$, as required.

7.3.5 Example: Suppose $f'(x)$ and $g'(x)$ exists, $g'(x) \neq 0$, and $f(x) = g(x) = 0$. Prove that

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(t)}{g'(t)}.$$

(This holds also for complex functions.)

Solution: Since $f(x) = g(x) = 0$, we have

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \lim_{t \to x} \frac{\frac{f(t)-f(x)}{t-x}}{\frac{g(t)-g(x)}{t-x}}$$

$$= \frac{\lim_{t \to x} \frac{f(t)-f(x)}{t-x}}{\lim_{t \to x} \frac{g(t)-g(x)}{t-x}}$$

$$= \frac{f'(t)}{g'(t)}.$$

7.3.6 Example: Suppose f' is continuous on $[a, b]$ and $\epsilon > 0$. Prove that there exists $\delta > 0$ such that

$$= \left|\frac{f(t) - f(x)}{t - x} - f'(x)\right| < \epsilon$$

whenever $0 < |t - x| < \delta$, $a \le x \le b$, $a \le t \le b$. (This could be expressed by saying that f is uniformly differentiable on $[a, b]$ if f' is continuous on $[a, b]$.) Dose this hold for vector-valued functions too?

Solution: let δ be such that $|f'(x) - f'(u)| < \epsilon$ for all $x, u \in [a, b]$ with $|x - u| < \delta$.

Then if $0 < |t - x| < \delta$ there exists u between t and x such that

$$= \frac{f(t) - f(x)}{t - x} = f'(u),$$

and hence, since $|u - x| < \delta$,

$$= \left|\frac{f(t) - f(x)}{t - x} - f'(x)\right|$$

$$= |f'(u) - f'(x)| < \epsilon.$$

Since this results holds for each component of a vector-valued function $\mathbf{f}(x)$, it must holds also for $\mathbf{f}$.

7.3.7 Example: Give an example a continuous function which is not differentiable.

Solution: Let f be defined by

$$f(x) = \begin{cases} x\sin\frac{1}{x} & for\ x \neq 0 \\ 0 & for\ x = 0 \end{cases}$$

Now we prove that f is continuous at $x = 0$ but not differentiable at $x = 0$.

R. H. L $\lim_{x \to 0+} f(x) = \lim_{x \to 0+0}\left(x\sin\frac{1}{x}\right)$

$= \lim_{h \to 0}\left(h\sin\frac{1}{h}\right)$ where $x = 0 + h$

$= \lim_{h \to 0}(h) \lim_{h \to 0}\left(\sin\frac{1}{h}\right)$

$= 0 \times$ (a finite quantity between -1 and 1)

$= 0$

L. H. L $\lim_{x \to 0-} f(x) = \lim_{x \to 0-0}\left(x\sin\frac{1}{x}\right)$

$= \lim_{h \to 0}\left((0 - h)\sin\frac{1}{0-h}\right)$ where $x = 0 - h$

$= \lim_{h \to 0}(h) \lim_{h \to 0}\left(\sin\frac{1}{h}\right)$

$= 0 \times$ (a finite quantity between -1 and 1)

$= 0$

So R. H. L= L. H. L = 0

$\therefore \lim_{x \to 0} f(x) = 0$

Also $f(0) = 0$

$\Rightarrow \lim_{x \to 0} f(x) = f(0)$

This shows that f is continuous at $x = 0$.

Differentiability:

L. H. D = $f'(0-0) = \lim_{t \to 0-0} \frac{f(t)-f(0)}{t-0}$

$= \lim_{t \to 0-0} \frac{t\sin\frac{1}{t}-0}{t-0}$

$= \lim_{t \to 0-0} \sin\frac{1}{t}$

$= \lim_{h \to 0} \sin\frac{1}{-h}$ where $t = 0 - h$

$= \lim_{h \to 0} \left((-1)\sin\frac{1}{h}\right)$

Which does not exist.

R. H. D = $f'(0+0) = \lim_{t \to 0+0} \frac{f(t)-f(0)}{t-0}$

$= \lim_{t \to 0+0} \frac{t\sin\frac{1}{t}-0}{t-0}$

$= \lim_{t \to 0+0} \sin\frac{1}{t}$

$= \lim_{h \to 0} \sin\frac{1}{h}$

Which does not exist.
Since Neither the left hand derivative nor the right hand derivative exists at $x = 0$.
f has no derivative at $x = 0$.

Hence f is not differentiable at $x = 0$.

Exercise :

1. Define derivative of a function on $[a, b]$.
2. Show that sum of two differentiable functions is differentiable.
3. Give an example a continuous function which is not differentiable.
4. State and prove chain rule.

7.4 SUMMARY:

This comprehensive lesson introduces learners to the concept of derivatives of real functions, providing a solid foundation for optimization techniques. Through a combination of theoretical explanations, named theorems, and practical examples,

learners will develop problem-solving skills using derivatives to optimize functions. This lessons covers 1. Introduction to derivatives of real functions, Named theorems, such as The Chain Rule Theorem, Examples with solutions to illustrate key concepts, and Exercise problems to reinforce understanding and develop problem-solving skills.

7.5 TECHNICAL TERMS:

- ❖ Differentiable function
- ❖ Right hand limit
- ❖ Left hand limit
- ❖ Continuous
- ❖ Inverse function
- ❖ Strictly increasing
- ❖ Chain Rule

7.6 SELF ASSESSMENT QUESTIONS:

1. If $C_0 + \frac{C_1}{2} + \cdots\cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$,where $C_0, \cdots\cdots\cdots\cdots, C_n$ are real constants, prove that the equation $C_0 + C_1 x + \cdots\cdots\cdots + C_{n-1}x^{n-1} + C_n x^n = 0$ has at least one real root between 0 and 1.

2. Let f be defined on $[a, b]$. If f is differentiable at a point $x \subset [a, b]$, then f is continuous at x. Is the converse true? Justify Your Answer.

3. State and Prove Chain Rule.

7.7 SUGGESTED READINGS:

1. Principles of Mathematical Analysis, Third Edition, Mc Graw-Hill International Editions Walter Rudin.

2. Mathematical Analysis by Tom M. Apostol, Narosa Publishing House, 2nd Edition, 1985.

Dr. V. Amarendra Babu.

LESSON-8

MEAN VALUE THEOREMS AND THE CONTINUITY OF DERIVATIVES

OBJECTIVES:

The objective of the lesson is to help the learners to understand

- ❖ To understand and apply mean value theorems for real functions.
- ❖ To analyze continuity and differentiability of functions using mean value theorems.

STRUCTURE:

8.0 INTRODUCTION:

In this lesson we derived local maximum and local minimum and proved generalized mean vale theorem (Cauchy value theorem), Lagrange language mean value theorem and Darboux theorem.

8.1 MEAN VALUE THEOREMS:

8.1.1 Definition: Let f be a real valued function defined on a metric space x. we say that f has a local maximum at a point $p \in x$ if there exists $\delta > 0$ such that $f(q) \leq f(p)$, for all $q \in x$ with $d(p,q) < \delta$. We say that f has a local minimum at a point $q \in x$, if there exists $\delta > 0$ such that $f(q) \geq f(p)$ for all $q \in x$ with $d(p,q) < \delta$.

8.1.2 Theorem: Let f be defined on $[a,b]$; if f has a local maximum at a point x (a,b) and $f(x)$ exists.

Then $\exists$ $\delta > 0 \ni a < x - \delta < x < x + \delta < b$ and

$f(q) \leq f(x)$, $\forall q \in [a,b]$ with $d(x,q) < \delta$............*(1)*

let $t \in (x - \delta, x)$

Then $d(x,t) < \delta$

so, $f(t) \leq f(x)$ and $f(t) - f(x) \leq 0$.

So, $\frac{f(t)-f(x)}{t-x} > 0$, since $t - x < 0$.

$$\Rightarrow \lim_{6\to x}\left[\frac{f(t)-f(x)}{t-x}\right] \geq 0$$

$\therefore f'(x) \geq 0$……………………*(2)*

let $t \in (x, x + \delta)$

Then $d(x,t) < \delta$

So from (1), $f(t) \leq f(x)$

So, $\frac{f(t)-f(x)}{t-x} \leq 0$, since $t - x > x$

$$\Rightarrow \lim_{x\to\infty}\left[\frac{f(t)-f(x)}{t-x}\right] \leq 0$$

$\therefore f'(x) \leq 0$……………………*(3)*

From (2) & (3), $f'(x) = 0$.

Note: A similar result holds for local minimum.

8.1.3 Theorem: Let f and g are continuous real functions on $[a, b)$ which are differentiable is $[a, b]$, then there is a point $x \in (a, b)$ at which

$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x)$

Proof: Suppose that f and g are real continuous functions on $[a, b]$ and differentiable in (a, b).

Put $h(t) = [f(b) - f(a)]g'(t) - [g(b) - g(a)]f'(t)$, $(a \leq t \leq b)$………*(1)*

Then h is continuous on $[a, b]$, h is differentiable in (a, b), since f & g are continuous on $[a, b]$ and f, g are differentiable on (a, b).

Also $h(a) = [f(b) - f(a)]g(a) - [g(b) - g(a)]f(a)$

$= f(b)f(a) - g(b)g(a)$

and $h(b) = [f(b) - f(a)]g(a) - [g(b) - g(a)]f(b) = f(b)f(a) - g(b)g(a)$

$\therefore h(a) = h(b)$

Now, we show that $h'(x) = 0$ for some $x \in (a, b)$

Case I: Suppose that h is a constant function then,

clearly $h'(x) \quad \forall\, h(t) \neq h(a)$

Case II: Suppose that h is not a constant function.

Then $\exists\, t \in (a, b) \ni h(t) \neq h(a)$

Then either $h(t) > h(a)$ or $h(t) < h(a)$suppose that $h(t) > h(a)$

$\therefore$ By the well known theorem, h attains its maximum at some point $x \in [a, b]$

$\therefore h(t) \leq h(x)$, $t \in [a, b]$

Then h has local maximum at the point $x \in [a,b]$

$\therefore$ By the Theorem 8.1.2 , $h'(x) = 0$

If $h(t) < h(a)$, the same argument is the above, $\exists$ a point $x \in [a,b] \ni h'(x) = 0$ From (1)

$h'(x) = (f(b)-f(a))g'(x)-(g(b)-g(a))f'(x) = 0$

$\Rightarrow (f(b)-f(a))g'(x) = (g(b)-g(a))f'(x)$

8.1.4 : Theorem (Lagrange Mean value theorem) :

If f is a real continuous function on $[a,b]$, which is differential in (a,b), then there is a point $x \in [a,b]$ such that $f(b)-f(a) = (b-a)f'(x)$

Proof: Let f be a continuous real function on $[a,b]$ and differentiable in (a,b).

Put $g(x) = x$

Then g is continuous on $[a,b]$ & differentiable on (a,b) and $g'(x) = 1$.

Then by Theorem 8.1.3, there exists a point $x \in [a,b]$ such that $(f(b)-f(a))g'(x)-(g(b)-g(a))f'(x)$

so, $f(b)-f(a) = (b-a)f'(x)$.

8.1.5 :Theorem: Suppose that f is differentiable in (a,b)

(a) If $f'(x) \geq 0$ for all $x \in [a,b]$, then f is monotonically increasing

(b) If $f'(x) = 0$ for all $x \in [a,b]$, then f is constant.

(c) If $f'(x) \leq 0$ for all $x \in [a,b]$, then f is monotonically decreasing:

Proof: Suppose that f is differentiable is (a,b)

(a) Suppose that $f'(x) \geq 0$, for all $x \in [a,b]$

Let $x_1, x_2 \in [a,b]$ such that $x_1 < x_2$

clearly f is continuous on $[x_1,x_2]$ and differential in (x_1,x_2) so, by mean value theorem, $\exists\, x \in (x_1,x_2)$ such that

$f(x_2) \quad f(x_1) = x_2 \quad x_1 f'(x)$

$\Rightarrow \frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(x)$……………………..(1)

$\Rightarrow f(x_2) - f(x_1) \geq 0 \; (\because f'(x) \geq 0)$

$\Rightarrow f(x_1) \leq f(x_2)$ for $x_1 < x_2$

Hence f is monotonically increasing.

(b) Suppose that $f'(x) = 0$, for all $x \in [a,b]$.

So, from (1), $\frac{f(x_2)-f(x_1)}{x_2-x_1} = 0$

$\Rightarrow f(x_2) - f(x_1) = 0$

$\Rightarrow f(x_1) = f(x_2), \; \forall\, x_1, x_2 \in (a,b)$

$\therefore f$ is constant.

(c) Suppose that $f'(x) \leq 0, \forall\, x \in (a,b)$ suppose that $x_1 < x_2$

From (1), $\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(x) \leq 0$

$\Rightarrow f(x_2) - f(x_1) \leq 0$

$\Rightarrow f(x_1) \geq f(x_2)$

So, $x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2)$

Hence f is monotonically decreasing.

Note: We have already known that a function f may have a derivative f' which exists at every point, but is discontinuous at some point. How function has derivative.

In particular, derivatives which does not exist at every point of an interval has one important property is common with functions which are continuous on an interval. The precise statement follows.

8.1.6 Theorem: Suppose f is real differentiable function on $[a,b]$ and suppose $f'(a) < \lambda < f'(b)$. Then there is a point $x \in (a,b)$ such that $f'(x) = \lambda$.

A similar result holds if $f'(a) > f'(b)$.

Proof: Suppose that f is a real differentiable function on $[a,b]$ suppose $f'(a) < \lambda < f'(b)$.

Define a function g by $g(t) = f(t) - \lambda t$, $t \in (a,b)$

clearly g is differentiable on $[a,b]$. Since f is differentiable.

Here $g'(t) = f'(t) - \lambda$

Now $g'(a) = f'(a) - \lambda < 0$ and $g'(b) = f'(b) - \lambda > 0$

since $g'(a) < 0$, g is decreasing at 'a'

so, there exists, $t_1 \in (a,b)\ \ni g(a) > g(t_1)$

$\Rightarrow$ g has local maximum at some point $x \neq a$.

since $g'(a) > 0$, g is increasing at 'b'

Then there exists some $t_2 \in (a,b) \ni g(t_2) < g(b)$

$\Rightarrow \min\{g(x) | x \in (a,b)\} \leq g(t_2) < g(b)$

$\Rightarrow x \neq b$

so, $x \subset (a,b)$

$\Rightarrow g'(x) = 0$ since g has local maximum at $x \in (a,b)$

so, $f'(x) \quad \lambda = 0$.

Therefore $f'(a) = \lambda$

8.1.7 : Corollary: If f is differentiable function in $[a,b]$, then f' cannot have any simple discontinuous on $[a,b]$.

8.1.8 : Examples:

(1) If $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ for $x \neq 0$, $f(0) = 0$, show that f is continuous and differentiable everywhere and that $f'(0) = 0$. Further that f' has a discontinuity of second kind of the origin.

Solution: Since $f(0+0) = \lim_{h \to 0} (0+h)^2 \sin\left(\frac{1}{0+h}\right)$

$= \lim_{h \to 0} h^2 \sin\left(\frac{1}{h}\right) = 0$

$\Rightarrow f(0+0) = 0$

Similarly $f(0-0) = 0$

$\therefore f(0+0) = f(0-0) = 0 = f(0)$

$\Rightarrow f$ is continuous at $x = 0$.

Again consider $f'(0) = \lim_{h \to 0} h^2 \frac{(0+h)^2 \sin\left(\frac{1}{0+h}\right)}{h}$

$= \lim_{h \to 0} h \sin\left(\frac{1}{h}\right)$

$\therefore f$ is differentiable at $x = 0$

At all other points, it is easy to prove f is continuous and differentiable.

Now $f'(x) = f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$ at $x \neq 0$ and $f'(0) = 0$

Therefore $f'(0+0) = \lim_{h \to 0} \frac{2h \sin\left(\frac{1}{h}\right) - \cos\left(\frac{1}{h}\right)}{h}$

which does not exist. Similarly $f'(0-0)$ does not exist

Hence f'has a discontinuity of second kind at the origin.

(2) Prove that the function $f(x) = |x|$ is continuous at $x = 0$, but not differentiable at $x = 0$.

Sol: Since $f(0) = 0, f(0+0) = \lim_{h \to 0} f(0+h) = 0$

$= \lim_{h \to 0} |0+h| = 0$

and $f|0-0| = \lim_{h \to 0} f|0-h| = 0$

$= \lim_{h \to 0} |0-h| = 0$

Hence, f is continuous at $x = 0$

since $f'(0+0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h| - 0}{h} = 1$

and $f'(0-0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{h} = \lim_{h \to 0} \frac{|-h| - 0}{-h} = 1$

Therefore $f'(0+0) \neq f'(0-0)$

Hence f is not differentiable at $x = 0$.

8.2 SOME MORE EXAMPES WITH SOLUTIONS:

8.2.1 Example: If in the Cauchy's mean value theorem, we write

$$\phi(x) = e^x \text{ and } f(x) = e^{-x},$$

Show that 'c' is the arithmetic mean between a and b.

Solution: Hence,

$$\frac{\phi(b)-\phi(a)}{f(b)-f(a)}=\frac{e^b-e^a}{e^{-b}-e^{-a}}$$

$$=-e^a e^b$$

$$=-e^{a+b}.$$

and

$$\frac{\phi'(x)}{f'(x)}=\frac{e^x}{-e^{-x}}.$$

Therefore,

$$\frac{\phi'(c)}{f'(c)}=\frac{e^c}{-e^{-c}}$$

$$=-e^{2c}.$$

Substituting these values in Cauchy's mean value theorem, we get

$$=-e^{a+b}=-e^{2c}$$

or,

$$2c=a+b,$$

i.e., $c=\frac{1}{2}(a+b)$.

Hence 'c' is the arithmetic mean between a and b.

8.2.2 Example: Verify Cauchy's mean value theorem for the functions x^2 and x^3 in the interval $[1,2]$.

Solution: Let $f(x)=x^2$, $\phi(x)=x^3$. Then

$$\frac{\phi(2)-\phi(1)}{f(2)-f(1)}=\frac{8-1}{4-1}$$

$$=\frac{7}{3}$$

and

$$\frac{\phi'(x)}{f'(x)}=\frac{3x^2}{2x}$$

$$=\frac{3}{2}x.$$

Since,

$$\frac{\phi'(c)}{f'(c)}=\frac{3c^2}{2c}$$

$$=\frac{3}{2}c.$$

Therefore,

$$\frac{3}{2}c=\frac{7}{3}$$

or,

$$c = \frac{14}{9}.$$

Since this value of 'c' lies in interval $]1,\ 2[$, Cauchy's mean value theorem is verified.

8.2.3 Example: If, in the Cauchy's mean value theorem, we write

$$\phi(x) = \sqrt{x} \text{ and } f(x) = \frac{1}{\sqrt{x}},$$

then, c, is the geometric mean between a and b and if we write

$$\phi(x) = \left(\frac{1}{x^2}\right) \text{ and } f(x) = \left(\frac{1}{x}\right),$$

then, c, is the harmonic mean between a and d.

Solution: When $\phi(x) = \sqrt{x}$ and $f(x) = \left(\frac{1}{\sqrt{x}}\right)$, we have

$$\frac{\phi(b) - \phi(a)}{f(b) - f(a)} = \frac{\phi'(c)}{f'(c)}$$

$$\Rightarrow \frac{\sqrt{b} - \sqrt{a}}{\left(\frac{1}{\sqrt{b}}\right) - \left(\frac{1}{\sqrt{u}}\right)} = \frac{\left(\frac{1}{2}\right)c^{-\frac{1}{2}}}{-\frac{1}{2}c^{-\frac{3}{2}}}.$$

Thus

$$-\sqrt{(ab)} = -c \text{ or } c = \sqrt{(ab)},$$

That is, c is the geometric mean between a and b.

And when $\phi(x) = \left(\frac{1}{x^2}\right)$, $f(x) = \left(\frac{1}{x}\right)$, we have

$$\frac{\phi(b) - \phi(a)}{f(b) - f(a)} = \frac{\phi'(c)}{f'(c)}$$

$$\Rightarrow \frac{\frac{1}{b^2} - \frac{1}{a^2}}{\left(\frac{1}{b}\right) - \left(\frac{1}{a}\right)} = \frac{-c^{-3}}{-c^{-2}}.$$

Thus

$$\frac{a+b}{ab} = \frac{2}{c} \quad or \quad c = \frac{2ab}{a+b},$$

That is, c is the harmonic mean between a and b.

8.2.4 Example: Use Cauchy's mean value theorem to evaluate

$$\lim_{x \to 1}\left[\frac{\cos\frac{1}{2}\pi x}{\log\left(\frac{1}{x}\right)}\right].$$

Solution: Let $f(x) = \cos\left(\frac{1}{2}\pi x\right)$, $g(x) = \log x$, $a = x$, $b = 1$.

Putting these values in Cauchy's mean value theorem,

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}, a < c < b$$

We get

$$\frac{\cos\frac{1}{2}\pi - \cos\frac{1}{2}\pi x}{\log 1 - \log x} = \frac{-\frac{1}{2}\pi \sin\left(\frac{1}{2}\pi c\right)}{\frac{1}{c}}, x < c < 1\cdot$$

Taking limits as $x \to 1$ which implies that $c \to 1$, we get

$$\lim_{x\to 1}\left\{\frac{0 - \cos\left(\frac{1}{2}\pi x\right)}{\log\left(\frac{1}{x}\right)}\right\}$$

$$= \lim_{c\to 1}\left\{\frac{-\frac{1}{2}\pi \sin\left(\frac{1}{2}\pi c\right)}{\frac{1}{c}}\right\}$$

or

$$\lim_{x\to 1}\left\{\frac{-\cos\left(\frac{1}{2}\pi x\right)}{\log\left(\frac{1}{x}\right)}\right\} = -\frac{1}{2}\pi$$

as $\sin\left(\frac{1}{2}\pi c\right) \to 1$ as $c \to 1$

or

$$\lim_{x\to 1}\left\{\frac{\cos\left(\frac{1}{2}\pi x\right)}{\log\left(\frac{1}{x}\right)}\right\} = \frac{\pi}{2}\cdot$$

8.2.5 Example: Use Lagrange's mean value theorem to prove that
$1 + x < e^x < 1 + xe^x, \quad \forall x > 0.$

Solution: Consider the function
$f(x) = e^x$ in $[0,x]\cdot$
Then f is continuous in $[0,x]$ and differentiable in $]0, x[$.
Consequently by Lagrange's mean value theorem there exists $c \in]0, x[$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0}$$

or

$$e^c = \frac{e^x - 1}{x} \quad \cdots\cdots\cdots\cdots (1)$$

Now, $0 < c < x \Rightarrow e^0 < e^c < e^x \cdots\cdots\cdots\cdots\cdots\cdots (2)$
From (1) and (2),

$$e^0 < \frac{e^x - 1}{x} < e^x \quad \forall x > 0$$

or

$$1 < \frac{e^x - 1}{x} < e^x$$

or

$$x < e^x - 1 < xe^x$$

or

$$1 + x < e^x < 1 + xe^x,$$

Which proves the required result.

8.2.6 Example . Assuming the derivatives which occur are continuous, apply the mean value theorem to prove that
$\phi'(x) = F'\{f(x)\}\, f'(x), \quad where\ \phi(x) = F\{f(x)\}\cdot$

Solution: Let $f(x) = t$ so that $\phi(x) = f(t)$.
Now,

$$\phi'(x) = \lim_{h \to 0} \frac{\phi(x+h) - \phi(x)}{h}$$

$$= \lim_{h \to 0} \frac{F[f(x+h)] - F[f(x)]}{h}$$

$$= \lim_{h \to 0} \frac{F[f(x) + hf'(x+\theta_1 h)] - F[f(x)]}{h}, [0 < \theta_1 < 1]$$

[since $f(x+h) = f(x) + hf'(x+\theta_1 h)$, by the mean value theorem]

$$= \lim_{h \to 0} \frac{F(t+H) - F(t)}{h}, \text{where } H = hf'(x+\theta_1 h)$$

$$= \lim_{h \to 0} \frac{F(t) + HF'(t+\theta_1 H) - F(t)}{h}, [0 < \theta_2 < 1]$$

[since $F(t+H) = F(t) + HF'(t+\theta_2 H)$, by the mean value theorem]

$$= \lim_{h \to 0} \frac{HF'(t+\theta_2 H)}{h}$$

$$= \lim_{h \to 0} \frac{hf'(x+\theta_1 h) + F'[t+\theta_2 hf'(x+\theta_1 h)]}{h}$$

$$= f'(x)F'(t) = F'[f(x)]\, f'(x).$$

Note: This example provides an alternative proof of the Chain Rule(Lesson-7, Chain Rule Theorem).

8.3 SUMMARY:

This lesson provides a comprehensive exploration of Mean Value Theorems, empowering learners to analyze and understand real functions. Through a combination of theoretical foundations, proof-based explanations, and illustrative examples, learners will develop expertise in applying Mean Value Theorems to investigate continuity and differentiability.

8.4 TECHNICAL TERMS:

- Metric Space
- Local Maximum
- Local Minimum
- Continuous function
- Differentiable function
- Constant function
- Monotonically Increasing

- Monotonically Decreasing
- Discontinuous
- Discontinuity of Second kind
- Arithmetic mean
- Geometric mean
- Harmonic mean
- Origin

8.5 SELF ASSESSMENT QUESTIONS

1. Let f and g are continuous real functions on $[a,b]$ which are differentiable is $[a,b]$, then there is a point $x \in (a,b)$ at which $[f(b)-f(a)]g'(x) = [g(b)-g(a)]f'(x)$.

2. Suppose that f is differentiable in (a,b)

(a) If $f'(x) \geq 0$ for all $x \in [a,b]$, then f is monotonically increasing

(b) If $f'(x) = 0$ for all $x \in [a,b]$, then f is constant.

(c) If $f'(x) \leq 0$ for all $x \in [a,b]$, then f is monotonically decreasing.

3. If $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ for $x \neq 0$, $f(0) = 0$, show that f is continuous and differentiable everywhere and that $f'(0) = 0$. Further that f' has a discontinuity of second kind of the origin.

4. Assuming the derivatives which occur are continuous, apply the mean value theorem to prove that $\phi'(x) = F'\{f(x)\}f'(x)$, $where\ \phi(x) = F\{f(x)\}$.

8.6 SUGGESTED READINGS:

1. Principles of Mathematical Analysis, Third Edition, Mc Graw-Hill International Editions Walter Rudin.

2. Mathematical Analysis by Tom M. Apostol, Narosa Publishing House, 2nd Edition, 1985.

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LESSON-9

L'HOSPITAL'S RULE AND DERIVATIVES OF HIGHER ORDER, TAYLOR'S THEOREM

OBJECTIVES:

The objective of the lesson is to help the learners to understand

- ❖ To study another equally important concept namely differentiation that is essential in the study of velocity and acceleration of continues paths.
- ❖ To analyze L-hospital's rule and derivatives of higher order, Taylor's Theorem.

STRECTURE:

9.1 INTRODUCTION :

In this lesson, we introduce higher order derivative and proved two theorems L-Hospital's rule and Taylor's theorems.

L'HOSPITAL'S RULE USES:

Using L Hospital's rule, we can solve the problem in $0/0$, ∞/∞, $\infty - \infty$, $0 \times \infty$, 1^{∞}, ∞^{0}, or 0^{0} forms. These forms are known as indeterminate forms. To remove the indeterminate forms in the problem, we can use L'Hospital's rule.

9.1.1. L-Hospital rule theorem : Suppose that f and g are real and differentiable in (a, b) $g'(x) \neq 0$, for all $x \in (a, b)$ where $-\infty \leq a \leq b \leq +\infty$ suppose

(1) $\frac{f'(x)}{g'(x)} \to A$ as $x \to a$

if

(2) $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$

or if

(3) $g(x) \to +\infty$ as $x \to a$

then

(4) $\frac{f(x)}{g(x)} \to A$ as $x \to a$

Proof: To prove this theorem in two cases

(1) $-\infty \leq A < +\infty$ and $-\infty < A \leq +\infty$

Case I: Suppose $-\infty \leq A < +\infty$

Choose a real number q such that $A < q$ then choose r such that $A < r < q$.

since $\frac{f'(x)}{g'(x)} \to A$ as $x \to a$, there is a point $c \in (a, b)$ such that

$\frac{f(x)}{g(x)} < A$ if $a < x < y < c$............................*(5)*

Then by, known theorem, there is a point $t \in (x, y)$, such that

$(f(y) - f(x))g'(t) = [g(y) - g(x)]f'(t)$

So, $\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(t)}{g'(t)} < r$...........................*(6)*

Suppose (2) holds, Then by (6)

$\frac{f(y)}{g(y)} = \frac{f'(t)}{g'(t)} < r$ $(\because a < x < t < y < c)$

$\Rightarrow \frac{f(y)}{g(y)} \leq r < q$ if $a < y < c$.....................*(7)*

then there exists a point $c_1 \in (a, y)$ such that $g(x) > g(y)$ and $g(x) > 0$, if $a < x < c_1$

Now multiplying *(6)* by $\frac{g(y)-g(x)}{g(x)}$ on both sides

$$\frac{f(y) - f(x)}{g(y) - g(x)} \cdot \frac{g(y) - g(x)}{g(x)} = \frac{f'(t)}{g'(t)}\left[\frac{g(y) - g(x)}{g(x)}\right] < r\left[\frac{g(y) - g(x)}{g(x)}\right]$$

$$\Rightarrow \frac{f(y) - f(x)}{g(x)} = \frac{f'(t)}{g'(t)}\left[\frac{g(y) - g(x)}{g(x)}\right] < r\left[\frac{g(y) - g(x)}{g(x)}\right]$$

$$\Rightarrow \frac{f(y) - f(x)}{g(x)} = \frac{f'(t)}{g'(t)}\left[\frac{g(x) - g(y)}{g(x)}\right] < r\left[1 - \frac{g(y)}{g(x)}\right]$$

$$\Rightarrow \frac{f(x)}{g(x)} - \frac{f(y)}{g(x)} < r\left[1 - \frac{g(y)}{g(x)}\right]$$

$$\Rightarrow \frac{f(x)}{g(x)} < r < r\frac{g(y)}{g(x)} + \frac{f(y)}{g(x)} \text{ .... } (8)$$

Since $g(x) \to +\infty$ as $x \to a$, taking limits on both sides exists a point $c_2 \in (a_1, c_1)$ such that

$\Rightarrow \dfrac{f(x)}{g(x)} < r$ if $a < x < c_2 < c_1$

$\Rightarrow \dfrac{f(x)}{g(x)} \le r < q, \qquad a < x < c_2 < c_1$ ……………… (9)

Case II: Suppose $-\infty < A \le +\infty$

choose p such that $p < A$

By same argument in **case I** there is a point $c \ni$

$p < \dfrac{f(x)}{g(x)} (a < x < c_3)$ ……………………… (10)

$\therefore \dfrac{f(x)}{g(x)} \to A \;\; as \;\; x \to a \cdot$

9.2 DERIVATIVES OF HIGHER ORDER TAYLOR'S THEOREM :

9.2.1. Definition: If f has a derivativesf'on an interval and if f' is itself differentiable we denote the derivative on f' by f'' and call the second derivatives of f, continuing in this manner, we obtain functions $f, f', f'', f''', \ldots, f^{(n)}$ each of which is derivative of the proceeding $f^{(n)}$ is called the n^{th} (or) derivative of order n, of f.

In order for $f'(x)$ to exists at a point x, $f^{(n-1)}(t)$ must be differentiable at x. Since $f^{(n-1)}$ must exist is a neighborhood of x. $f^{(n-2)}$ must be distinct point of $[a, b]$ and define.

9.2.2. Theorem (Taylor's Theorem):

Suppose f is a real function on $[a, b]$, n is a positive integer, $f^{(n-1)}$ is continuous on $[a, b]$, $f^{(n)}(t)$ exists for away $t \in (a, b)$. Let α, β be distinct points of $[a, b]$ and define

$$P(t) = \sum_{k=0}^{n} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Then there exists a point x between α and β and such that

$$f(\beta) = p(\beta) + \frac{f^{(n-1)}(x)}{n!}(\beta-\alpha)^n \; \ldots\ldots\ldots\ldots (2)$$

Proof: If $n = 1$, the Taylor's reduces to mean value theorem suppose that $n > 1$

Let M be a number defined by

$f(\beta) = p(\beta) + M(\beta - \alpha)^n$ …………………… (3)

and put

$g(t) = f(t) - P(t) - M(t-\alpha)^n \qquad (a \le t \le b)$ ……………… (4)

Now, we show that

(i) $P^{(k)}(\alpha) = f^{(k)}(\alpha)$, for $k = 0,1,2, \ldots (n-1)$ and

(ii) $g(\alpha) = g'(\alpha) = \cdots = g^{(n-1)}(\alpha) = 0$

From (1),

$$P(t) - f(\alpha) + \frac{f'(\alpha)}{1!}(t-\alpha) + \frac{f''(\alpha)}{2!}(t-\alpha)^2 + \cdots + \frac{f^{(n-1)}(\alpha)}{(n-1)!}(t-\alpha)^{n-1}$$

So, $P(\alpha) = f(\alpha)$

Now, $x_n \in (\alpha, x_{n-1})$

$\Rightarrow P'(\alpha) = 0 + f'(\alpha) + 0 + \cdots + 0$

$\Rightarrow P'(\alpha) = f'(\alpha)$

Now $P^{(n-1)}(t) = \frac{f^{(n-1)}(\alpha)}{(n-1)!} = f^{(n-1)}(\alpha)$

$\therefore P^{(k)}(\alpha) = f^{(k)}(\alpha)$, for $k = 0,1,2, \ldots .(n-1)$

Also, $g'(t) = f'(t) - P'(t) - M(t-\alpha)^n$ from (4)

So, $g(\alpha) = f(\alpha) - P(\alpha)$

$g(\alpha) = 0$ and $g'(t) = f'(t) - P'(t) - M(t-\alpha)^{n-1}$

$g'(\alpha) = f'(\alpha) - P'(\alpha) = 0$

similarly, we prove that $g''(\alpha) = g'''(\alpha) = \cdots = g^{(n-1)}(\alpha) = 0$

Now, $g^{(n)}(t) = f^{(n)}(t) - Mn!$

$\Rightarrow g^{(n)}(t) = f^{(n)}(t) - Mn!$ (degree of $P(t) = n-1$, so $P^{(n)}(t) = 0$)………(5)

Now from (4), $g(\beta) = f(\beta) - P(\beta) - M(\beta - \alpha)^n = 0$

from (3) we know g is continuous on $[\alpha, \beta]$ and differentiable in (α, β).

Then by mean value theorem, there exists some $x_1 \in (\alpha, \beta)$ such that

$g(\beta) - g(\alpha) = (\beta - \alpha)g'(x_1)$

$0 = (\beta - \alpha)g'(x_1)$

$\therefore g'(x_1) = 0$

Also g' is continuous on $[\alpha, x_1]$ and differentiable 0 in (α, x_1), again by mean value theorem there is some $x_2 \in (\alpha, x_1)$ such that

$g'(x_1) - g'(\alpha) = (x_1 - \alpha)g''(x_2)$

$\Rightarrow g''(x_2) = 0$

We continue this process, there exists a point $x_n \in (\alpha, x_{n-1})$ such that $g^{(n)}(x_n) = 0$

By from (5), $g^{(n)}(x_n) = f^{(n)}(x_n) - M \cdot n!$

$\Rightarrow f^{(n)}(x_n) = M \cdot n!$

$\therefore M = \frac{f^{(n)}(x_n)}{n!}$

Put $x = x_n$

Then $M = \frac{f^{(n)}(x)}{n!}$

Substituting in (3) we see

$f(\beta) = p(\beta) + \frac{f^{(n)}(x)}{n!}(\beta - a)^n.$

9.2.3. Theorem : (Taylor's theorem with Cauchy's form of remainder)

If f is a real valued function on $[a,\ a+h]$ such that all the derivatives upto $(n-1)th$ are continuous in $a \le x \le a+h$ and $f^{(n)}(x)$ exists in $a < x < a+h$, then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots$$

$$\cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{(n-1)!}(a-\theta)^{n-1}f^{(n)}(a+\theta h),$$

where $0 < \theta < 1$.

Proof: Consider the function ϕ defined by

$$\phi(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!}f''(x) + \cdots$$

$$\cdots + \frac{(a+h-x)^{n-1}}{(n-1)!}f^{(n-1)}(x) + (a+h-x)A,$$

where A is a constant so chosen that $\phi(a+h) = \phi(a)$,

i.e., $f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots$

$$\cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + hA \cdots\cdots\cdots (1)$$

It is easy to see that ϕ is differentiable in $]a,\ a+h[$. Hence ϕ satisfies all the conditions of Rolle's theorem.

Therefore, $\phi'(a+\theta h) = 0 \quad [0 < \theta < 1]$.

But

$$\theta'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!}f^{(n)}(x) - A,$$

Since other terms cancel in pairs.

Therefore, $0 = \phi'(a+0h) = \frac{(h-\theta h)^{n-1}}{(n-1)!}f^{(n)}(a+0h) - A$

or

$$A = \frac{h^{n-1}}{(n-1)!}(1-\theta)^{n-1}f^{(n)}(a+\theta h).$$

Substituting this value of A in (1), we get

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots\cdots$$

$$\cdots\cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{(n-1)!}(1-\theta)^{n-1}f^{(n)}(a+\theta h) \cdots\cdots\cdots (\boldsymbol{A})$$

The $(n+1)th$ term

$$\frac{h^n}{(n-1)!}(1-\theta)^{n-1}f^{(n)}(a+\theta h) \cdots\cdots\cdots\cdots (\boldsymbol{B})$$

is called Cauchy's form of remainder after n times in the Taylor's expansion of $f(a+h)$ in ascending integral powers of h.

9.2.4. Corollary: (Maclaurin's theorem with Cauchy's form of remainder)

If we change a to 0 and h to x in $(\boldsymbol{A})$, we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots\cdots$$

$$\cdots\cdots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{x^n}{(n-1)!}(1-\theta)^{n-1}f^{(n)}(\theta x), [0 < \theta < 1] \cdots\cdots\cdots (\boldsymbol{C})$$

The $(n+1)th$ term

$$\frac{x^n}{(n-1)!}(1-\theta)^{n-1}f^{(n)}(\theta x) \cdots\cdots\cdots\cdots (\boldsymbol{D})$$

is known as Cauchy's form of Remainder in Maclaurin's development of $f(x)$ in the interval $[0,\ x]$.

9.3. SOME MORE EXAMPLES WITH SOLUTIONS:

9.3.1. Example: Let f be a continuous real function on R', of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 3$ as $x \to 0$. Dose it follow that $f'(0)$ exists?

Solution: Yes,

By L'Hospital's rule

$$\lim_{t\to 0}\frac{f(t)-f(0)}{t} = \lim_{t\to 0} f'(t) = 3,$$

and this by definition means that $f'(0) = 3$.

9.3.2. Example : Evaluate $\lim_{t\to 0}\frac{(2\sin t - \sin 2t)}{t - \sin t}$

Solution: Given,

$$\lim_{t\to 0}\frac{(2\sin t - \sin 2t)}{t - \sin t}$$

Differentiate the above form, we get

$$= \lim_{t\to 0}\frac{(2\cos t - 2\cos 2t)}{1 - \cos t}$$

$$= \lim_{t \to 0} \frac{(-2\sin t + 4\sin 2t)}{\sin t}$$
$$= \lim_{t \to 0} \frac{(-2\cos t + 8\cos 2t)}{\cos t}$$

Now substitute the limit,

$$= \frac{-2+8}{1}$$

$$= \frac{6}{1} = 6$$

Therefore,

$$\lim_{t \to 0} \frac{(2\sin t - \sin 2t)}{t - \sin t} = 6.$$

9.3.3. Example: Evaluate $\lim_{t \to 0} \frac{\sin 3t}{\sin 4t}$

Solution: Given,

$$\lim_{t \to 0} \frac{\sin 3t}{\sin 4t}$$

$$= \lim_{t \to 0} \frac{3\cos 3t}{4\cos 4t}$$

Now substitute the limit,

$$= \frac{3\cos 0}{4\cos 0}$$

$$= \frac{3}{4}$$

Therefore,

$$\lim_{t \to 0} \frac{\sin 3t}{\sin 4t} = \frac{3}{4}.$$

9.3.4. Example: Prove that

$$\sin ax = ax - \frac{a^3x^3}{3!} + \frac{a^5x^5}{5!} - \cdots\cdots + \frac{a^{n-1}x^{n-1}}{(n-1)!}\sin\left(\frac{n-1}{2}\pi\right) + \frac{a^nx^n}{n!}\sin\left(a\theta x + \frac{n\pi}{2}\right).$$

Solution: Here $f(x) = \sin ax$.

Therefore, $f'(x) = a\cos ax = a\sin\left(ax + \frac{\pi}{2}\right)$

$f''(x) = a^2\cos\left(ax + \frac{\pi}{2}\right) = a^2\sin\left(ax + 2\cdot\frac{\pi}{2}\right)$

$f'''(x) = a^3\cos\left(ax + 2\cdot\frac{\pi}{2}\right) = a^3\sin\left(ax + 3\cdot\frac{\pi}{2}\right)$

$f''''(x) = a^4\cos\left(ax + 3\cdot\frac{\pi}{2}\right) = a^4\sin\left(ax + 4\cdot\frac{\pi}{2}\right)$

$f'''''(x) = a^5 \cos\left(ax + 4 \cdot \frac{\pi}{2}\right) = a^5 \sin\left(ax + 5 \cdot \frac{\pi}{2}\right)$

... ...

... ...

... ...

... ...

... ...

$f^{(n-1)}(x) = a^{n-1} \sin\left(ax + \frac{n-1}{2}\pi\right)$

and

$$f^{(n)}(x) = a^n \sin\left(ax + \frac{n\pi}{2}\right)$$

So that

$$f^{(n)}(\theta x) = a^n \sin\left(a\theta x + \frac{n\pi}{2}\right).$$

Therefore,

$$f(0) = \sin 0 = 0;$$

$$f'(0) = a \sin\frac{\pi}{2} = a;$$

$$f''(0) = a^2 \sin \pi = 0;$$

$$f'''(0) = a^3 \sin\frac{3\pi}{2} = -a^3;$$

$$f^{iv}(0) = a^4 \sin 2\pi = 0;$$

$$f^{v}(0) = a^5 \sin\frac{5\pi}{2} = a^5;$$

... ...

... ...

... ...

... ...

... ...

$$f^{(n-1)}(0) = a^{n-1} \sin\frac{n-1}{2}\pi \cdot$$

Substituting these values in

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \cdots$

$$\cdots\cdots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{x^n}{n!}f^{(n)}(\theta x),$$

We get,

$$\sin ax = 0 + xa + 0 - \frac{x^3}{3!}a^3 + 0 + \frac{x^5}{5!}a^5 + 0 - \cdots\cdots + \frac{x^{n-1}}{(n-1)!}a^{n-1}\sin\left(\frac{n-1}{2}\pi\right) + \frac{x^n}{n!}a^n \sin\left(a\theta x + \frac{n\pi}{2}\right)$$

or

$$\sin ax = ax - \frac{a^3x^3}{3!} + \frac{a^5x^5}{5!} - \cdots\cdots + \frac{a^{n-1}x^{n-1}}{(n-1)!}\sin\left(\frac{n-1}{2}\pi\right) + \frac{a^n x^n}{n!}\sin\left(a\theta x + \frac{n\pi}{2}\right).$$

Hence the solution of this example.

9.3.5. Example: Show that ' θ ' (which occurs in the Lagrange's mean value theorem) approaches the limit $\frac{1}{2}$ as ' h ' approaches zero provided that $f''(a)$ is not zero. It is assumed that $f''(x)$ is continuous.

Solution: Since $f''(x)$ is continuous at $x = 0$, it follows that $f''(a)$ exists.
Hence by Taylor's theorem, we get

$$f(a+h) = hf'(a) + \frac{h^2}{2!}f''(a+\theta' h) \cdots\cdots\cdots\cdots (1)$$

Also by mean value theorem, we have

$$f(a+h) = f(a) + hf'(a+\theta h) \cdots\cdots\cdots\cdots (2)$$

Substituting (2) from (1), we get

$$0 = hf'(a) + \frac{h^2}{2!}f''(a+\theta' h) - hf'(a+\theta h)$$

or

$$f'(a+\theta h) - f'(a) = \frac{h}{2}f''(a+\theta' h) \cdots\cdots\cdots\cdots\cdots (3)$$

Again since f' is continuous and differentiable, we have by mean value theorem,

$$f'(a+\theta h) = f'(a) + \theta h f''(a+\theta''\theta h)$$

$$f'(a+\theta h) - f'(a) = \theta h f''(a+\theta''\theta h) \cdots\cdots\cdots\cdots\cdots (4)$$

From (3) and (4), we get

$$\theta h f''(a+\theta''\theta h) = \frac{h}{2}f''(a+\theta' h)$$

or

$$\theta = \frac{1}{2}\frac{f''(a+\theta' h)}{f''(a+\theta''\theta' h)}.$$

Hence,

$$\lim_{h\to 0}\theta - \frac{1}{2}\frac{f''(a)}{f''(a)},$$

$$=\frac{1}{2},$$

provided $f''(a) \neq 0$.

9.3.6. Example: If $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(\theta x)$ find the value of ' θ ' as x tends to 1, $f(x)$ being $(1-x)^{\frac{5}{2}}$.

Solution: $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(\theta x)$ .... (1)

We have

$f(0) = 1, f'(0) = -\frac{5}{2}, f''(\theta x) = -\frac{15}{4}(1-\theta x)^{\frac{1}{2}}$.

Hence substituting these values in (1), we get

$(1-x)^{\frac{5}{2}} = 1 - \frac{5}{2}x + \frac{x^2}{2!} \times \frac{5}{2}(1-\theta x)^{\frac{1}{2}}$.

Therefore, as $x \to 1$, we get

$$0 = 1 - \frac{5}{2} + \frac{1}{2!} \cdot \frac{15}{4}(1-\theta)^{\frac{1}{2}}$$

or

$$(1-\theta)^{\frac{1}{2}} = \frac{4}{5}$$

or

$$(1-\theta) = \frac{16}{25}.$$

This gives $\theta = \frac{9}{25}$.

9.3.7. Example : If $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x+\theta h)$, find that value of ' θ ' as x tends to a, $f(x)$ being $(x-1)^{\frac{5}{2}}$.

Solution: $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x+\theta h)$ .... (1)

We have

$f(x+h) = f(x+h-a)^{\frac{5}{2}}, \quad f'(x) = \frac{5}{2}(x-a)^{\frac{3}{2}}$,

$f''(x+\theta h) = \frac{15}{4}(x+\theta h - a)^{\frac{1}{2}}$,

Substituting these values in (1), we get

$$(x+h-a)^{\frac{5}{2}} = (x-a)^{\frac{5}{2}} + \frac{5}{2}(x-a)^{\frac{3}{2}}h + \frac{15}{4}(x+\theta h - a)^{\frac{1}{2}}\frac{h^2}{2!} \text{ .... } (2)$$

Hence as $x \to a$, we get from (2),

$h^{\frac{5}{2}} = \frac{15}{4}(\theta h)^{\frac{1}{2}} \cdot \frac{h^2}{2!}$

or

$\theta = \frac{64}{225}$.

9.4 SUMMARY:

This lesson explores the fundamental concept of differentiation, crucial for understanding velocity and acceleration in continuous paths. Learners will delve into advanced calculus topics, including L'Hospital's Rule, higher-order derivatives, and Taylor's Theorem. This lesson covers - Introduction to differentiation and its applications, L'Hospital's Rule for indeterminate forms, Derivatives of higher order, Taylor's Theorem and its applications, Taylor's Theorem with Cauchy's form of remainder, Maclaurin's Theorem with Cauchy's form of remainder, and Practice examples with solutions to reinforce understanding.

9.5 TECHNICAL TERMS:

- Neighborhood
- Real Valued function
- Continuous
- Differentiable
- Higher order derivative
- Remainder

9.6 SELF ASSESSMENT QUESTIONS

1. State and Prove L' Hospital's Rule

2. State and Prove Taylor's Theorem.

3. Prove that

$$\sin ax = ax - \frac{a^3x^3}{3!} + \frac{a^5x^5}{5!} - \cdots\cdots + \frac{a^{n-1}x^{n-1}}{(n-1)!}\sin\left(\frac{n-1}{2}\pi\right) + \frac{a^nx^n}{n!}\sin\left(c\theta x + \frac{n\pi}{2}\right)$$

4. If $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(\theta x)$ find the value of ' θ ' as x tends to 1, $f(x)$ being $(1-x)^{\frac{5}{2}}$.

9.7 SUGGESTED READINGS:

1. Principles of Mathematical Analysis, Third Edition, Mc Graw-Hill International Editions Walter Rudin.

2. Mathematical Analysis by Tom M. Apostol, Narosa Publishing House, 2nd Edition, 1985.

Dr. V. Amarendra Babu.

LESSON -10

DIFFERENTIATION OF VECTOR VALUED FUNCTIONS

OBJECTIVES:

The objective of the lesson is to help the learners to understand

- ❖ To understand the concept of definitions and computation of derivatives of vector valued functions and their properties.
- ❖ To apply differentiations for solving problems in Physics, Engineering and Mathematics.

STRUCTURE:

10.0 DIFFERENTIATION OF VECTOR VALUED FUNCTIONS

10.0.1 Definition: Let complex valued functions f defined on $[a, b]$, and If f_1 and f_2 are the real and imaginary parts of f, that is

$f(t) - f_1(t) + i\ f_2(t)$, for $a \leq t \leq b$, where $f_1(t)$ and $f_2(t)$ are real then we clearly have

$f^{'}(t) = f_1^{'}(t) + i\ f_2^{'}(t)$……………………….(1)

Also, f is differentiable at x if and only if both f_1 and f_2 are differentiable at x. Passing to vector valued functions in general, i.e., to functions f which map $[a, b]$ into some $\mathbb{R}^k$, we may still apply definition 10.1 to define $f^{'}(x)$: for each t a point in $\mathbb{R}^k$, and the limit is taken with respect to the norm of $\mathbb{R}^k$. In other words, $f^{'}(x)$ is that point of $\mathbb{R}^k$, for which

$$\lim_{t \to x}\left[\frac{f(t) - f(x)}{t - x} - f^{'}(x)\right] = 0$$

$f^{'}$ is again a function with values in $\mathbb{R}$

if $f = (f_1, f_2, \dots, f_k)$, then $f^{'} = \left(f_1^{'}, f_2^{'}, \dots, f_k^{'}\right)$ and f is differentiable at a point x if and only if each of the functions $f_1, f_2, \dots, f_k$ is differentiable at x.

When we turn to the mean value theorem, however and to be of its consequences, namely

L-Hospital's rule, the situation changes given two examples will show that L-Hospital rule & mean value theorem fails for complex valued functions.

10.0.2 Example :

Define, for real X, $f(x) = e^{ix} = \cos x + i \sin x$..................(3)

Then $f(2\pi) - f(0) = 1 - 1 = 0$....................................(4)

But $f'(x) = i\, e^{ix}$......................................(5)

So, that $|f'(x)| = |i\, e^{ix}| = |i|\, |e^{ix}| = 1$, for all real x.

So, the mean value theorem fails to hold this case.

(OR)

Let $f: \mathbb{R} \to C : f(x) = e^{ix} = \cos x + i \sin x$.

Consider the interval $[0,\ 2\pi]$.

Then the function f is continuous and differentiable for all $x \in \mathbb{R}$ so that the conditions of the mean value theorem are satisfied in the interval. But

$f(2\pi) - f(0) = 1 - 1 = 0,$

whereas $f'(x) = i \cdot e^{ix}$ so that $|f'(x)| - 1$ for all $x \in \mathbb{R}$.

Hence the mean value theorem does not hold in this case.

10.0.3 Example : Define $f(x) = x$ and $g(x) = x + x^2 \cdot e^{\frac{1}{x^2}},$

for $x \in (0,1)$.....................(6)

$$\Rightarrow \lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{x}{x + x^2 \cdot e^{\frac{1}{x^2}}}$$

$$= \lim_{x\to 0} \frac{x}{x\left(1 + x \cdot e^{\frac{1}{x^2}}\right)}$$

$$= \lim_{x\to 0} \frac{x}{x\left(1 + x \cdot e^{\frac{1}{x^2}}\right)} = 1 \qquad \left(\because x \cdot e^{\frac{1}{x^2}} \to 0 \; as \; x \to 0\right)$$

Now,

$f'(x) = 1$

and

$$g'(x) = 1 + x^2 \cdot e^{\frac{1}{x^2}}(-2ix - 3) + e^{\frac{1}{x^2}}\, 2x$$

$$= 1 - \frac{2i}{x} e^{\frac{1}{x^2}} + 2x e^{\frac{1}{x^2}}$$

$= 1 + \left(2x - \frac{2i}{x}\right)e^{\frac{1}{x^2}} \ldots\ldots\ldots\ldots\ldots.(7)$

$\Rightarrow |g^{'}(x)| = 1 + \left(2x - \frac{2i}{x}\right)e^{\frac{1}{x^2}}$

$\geq \left|2x - \frac{2i}{x}\right| - 1 \geq \frac{2}{x} - 1 \ldots\ldots\ldots\ldots.(8)$

$= \left|\frac{f^{'}(x)}{g^{'}(x)}\right| = \frac{1}{|g^{'}(x)|} \leq \frac{x}{2-x} \ldots\ldots\ldots\ldots(9)$

$= \lim_{x\to 0} \frac{f^{'}(x)}{g^{'}(x)} = 0 \ldots\ldots\ldots\ldots\ldots\ldots(10)$

So L-Hospitals rule fails in this case

Note: By mean value theorem, it follows that

$|f(a) - f(b)| \leq (b-a) \operatorname*{lub}_{a<x<b} |f^{'}(x)|$

We shall now prove the vector-valued analogue of

$|f(a) - f(b)| \leq (b-a) \operatorname*{lub}_{a<x<b} |f^{'}(x)|$

In the below theorem.

10.0.4 Theorem : Suppose that f is a continuous mapping of $[a,b]$ into $\mathbb{R}^k$ and f is differentiable in (a,b). Then there exists $x \in (a,b)$ such that $|f(a) - f(b)| \leq (b-a)|f^{'}(x)|$.

Proof: Given that 'f' is a continuous mapping from $[a,b]$ in $\mathbb{R}^k$ and f is differentiable in (a,b).

Let $Z = f(b) - f(a)$ and

define $\phi(t) = zf(t), t \in [a,b]$

Then ϕ is real valued continuous function on $[a,b]$ which is differentiable in (a,b), since f is continuous on $[a,b]$ differentiable in (a,b)

Then, by mean value theorem, there exists $x \in (a,b)$ such that

$\phi(b) - \phi(a) = (b-a)\phi^{'}(x)$

$= (b-a)Zf^{'}(x)$……………….(1)

But $\phi(b) - \phi(a) = Zf(b) - Zf(a)$

$= Z(f(b) - f(a))$

$= Z \cdot Z = Z^2$…………………………….(2)

$\Rightarrow (b-a)Zf^{'}(x) = Z^2$………………………………….(3)

If $Z = 0$, then $f(b) - f(a) = 0$

$\Rightarrow 0 = |f(b) - f(a)| \leq (b-a)|f'(x)|$……………………………..(4)

Suppose that $Z \neq 0$, then from (3)

$|Z|^2 = |b-a||Zf'(x)| \leq (b-a)|Z| \cdot |f'(x)|$

So, $|Z|^2 \leq (b-a)|Z| \cdot |f'(x)|$

$|Z| \leq (b-a)|f'(x)|$

$\therefore |f(b) - f(a)| \leq (b-a)|f'(x)|$

10.1 PROPERTIES OF VECTOR VALUED FUNCTIONS:

All of the properties of differentiation still hold for vector values functions. Moreover because there are a variety of ways of defining multiplication, there is an abundance of product rules.

Suppose that $v(t)$ and $w(t)$ are vector valued functions, $f(t)$ is a scalar function, and c is a real number then

10.1.1 Property: $\frac{d}{dt}\left(v(t) + w(t)\right) = \frac{d}{dt}v(t) + \frac{d}{dt}w(t)$,

10.1.2 Property: $\frac{d}{dt}cv(t) = c\frac{d}{dt}v(t)$,

10.1.3 Property: $\frac{d}{dt}\left(f(t)v(t)\right) = f'(t)v(t) + f(t)v'(t)$,

10.1.4 Property: $\left(v(t) \cdot w(t)\right)' = v'(t) \cdot w(t) + v(t) \cdot w'(t)$,

10.1.5 Property: $\left(v(t) \times w(t)\right)' = v'(t) \times w(t) + v(t) \times w'(t)$,

10.1.6 Property: $\frac{d}{dt}v\left(f(t)\right) = v'(t)\left(f(t)\right)f'(t)$.

10.2 SOME MORE EXAMPLES WITH SOLUTION:

10.2.1 Example: Show that if r is a differentiable vector valued function with constant magnitude, then

$r \cdot r' = 0$.

Solution: Since r has constant magnitude, call its magnitude k,

$k^2 = |r|^2 = r \cdot r$.

Taking derivatives of the left and right sides gives

$0 = (r \cdot r)'$

$= r' \cdot r + r \cdot r'$

$= 2r \cdot r'$.

Divide by two and the result follows.

10.2.2 Theorem. Let $f: [a,b] \to \mathbb{R}^k$ and let f be differentiable at x_0 $(a < x_0 < b)$. If $a < \alpha_n < x_0 < \beta_n < b$ for $n = 1, 2, 3, \cdots\cdots$ and $\alpha_n \to x_0$, $\beta_n \to x_0$ as $n \to \infty$

then

$$\lim_{n\to\infty}\frac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n}=f'(x_0).$$

Proof: Let $\lambda_n=\frac{(\beta_n-x_0)}{(\beta_n-\alpha_n)}$ so that $0<\lambda_n<1$.

Then for each n, we have

$$\frac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n}-f'(x_0)$$

$$=\lambda_n\left\{\frac{f(\beta_n)-f(x_0)}{\beta_n-x_0}-f'(x_0)\right\}+(1-\lambda_n)\left\{\frac{f(\alpha_n)-f(x_0)}{\alpha_n-x_0}-f'(x_0)\right\}\text{..........}(1)$$

Since f is differentiable at x_0 the two expression within the brackets on the right hand side of (1) tend to 0 as $n\to\infty$ and since

$\langle\lambda_n\rangle$ and $\langle 1-\lambda_n\rangle$ are bounded sequences, it follows that the right hand side of (1) tends to 0 as $n\to\infty$.

Consequently, the left hand side of (1) also tends to 0 as $n\to\infty$, that is

$$\lim_{n\to\infty}\frac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n}=f'(x_0).$$

10.2.3 Example . Suppose f' is continuous on $[a,b]$ and $\epsilon>0$. Prove that there exists $\delta>0$ such that

$$\left|\frac{f(t)-f(x)}{t-x}-f'(x)\right|<\epsilon$$

Whenever $0<|t-x|<\delta, a\le x\le b, a\le t\le b.$

(This could be expressed by saying that f is uniformly differentiable on $[a,b]$ if f' is continuous on $[a,b]$.)

Does this hold for vector-valued functions too?

Solution. Let δ be such that $|f'(x)-f'(u)|<\epsilon$ for all $x,u\in[a,b]$ with $|x-u|<\delta.$

Then if $0<|t-x|<\delta$ there exists u between t and x such that

$$\frac{f(t)-f(x)}{t-x}=f'(u)$$

and hence, since $|u-x|<\delta$,

$$\left|\frac{f(t)-f(x)}{t-x}-f'(x)\right|$$

$$-|f'(u)-f'(x)|<\epsilon.$$

Since this result holds for each component of a vector-valued function $f(x)$, it must hold also for f.

10.2.4 Example . Let f be a continuous real function on R', of which it is known that $f'(x)$ exists for all $x\neq 0$ and that $f'(x)\to 3$ as $x\to 0$. Does it follow that $f'(0)$ exists?

Solution. Yes. By L'Hospital's rule

$$\lim_{t\to 0}\frac{f(t)\quad f(0)}{t}$$

$= \lim_{t\to 0} f'(t) = 3,$

And this by definition means that $f'(0) = 3$.

10.2.5 Example . Suppose f is defined in a neighbourhood of x, and suppose $f''(x)$ exists. Show that

$$\lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x)$$

Solution. For a real-valued function this is a routine application of L' Hospital's rule:

$$\lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}$$

$$= \lim_{h\to 0} \frac{f'(x+h) - f'(x-h)}{2h}$$

$$= \frac{1}{2} \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} + \frac{f'(x) - f'(x-h)}{h}$$

$$= f''(x)$$

For complex-valued functions the result follows from separate consideration of real and imaginary parts.

The limit will be zero at $x = 0$ for any odd function f whatsoever, even if the function is not continuous.

For example we could take $f(x) = sgn\ (x)$, which is $+1$ for $x > 0$, and -1 for $x < 0$.

10.2.6 Example . Suppose $a \in R^1$, f is a twice-differentiable real function on (a, ∞), and M_0, M_1, M_2 are the least upper bounds of $|f(x)|$, $|f'(x)|$, $|f''(x)|$ respectively, on (a, ∞). Prove that $M_1^2 \le 4M_0M_2$.

Hint: If $h > 0$, Taylor's theorem shows that

$$f'(x) = \frac{1}{2h}[f(x+2h) - f(x)] - hf''(\xi)$$

for some $\xi \in (x, x+2h)$. Hence

$$|f(x)| \le hM_2 + \frac{M_0}{h}$$

To show that $M_1^2 \le 4M_0M_2$ can actually happen, take $a = -1$, define

$$f(x) = \begin{cases} 2x^2 - 1, (-1 < x < 0), \\ \dfrac{x^2 - 1}{x^2 + 1}, \quad (0 \le x < \infty), \end{cases}$$

and show that $M_0 = 1, M_1 = 4, M_2 = 4$.

Does $M_1^2 \le 4M_1M_2$ hold for vector-valued functions too?

Solution. The inequality is obvious if $M_0 = +\infty$ or $M_2 = +\infty$,

So we shall assume that M_0 and M_2 are both finite.

We need to show that

$$|f'(x)| \le 2\sqrt{M_0M_2}$$

for all $x > a$. We note that this is obvious if $M_2 = 0$.

Since in that case $f'(x)$ is constant,

$f(x)$ is a linear function, and the only bounded linear function is a constant, whose derivative is zero.

Hence we shall assume from now on that $0 < M_2 < +\infty$ and $0 < M_0 < +\infty$.

Following the hint,

We need only choose $h = \sqrt{\frac{M_0}{M_2}}$, and we obtain

$|f'(x)| \leq 2\sqrt{M_0 M_2}$

Which is precisely the desired inequality.

The case of equality follows, since the example proposed satisfies

$$f(x) = 1 - \frac{2}{x^2+1} \quad \text{for } x \geq 0.$$

We see easily that $|f(x)| \leq 1$ for all $x > -1$.

Now,

$$f'(x) - \frac{4x}{(x^2+1)^2} \quad \text{for } x > 0$$

and

$f'(x) = 4x \;$ for $x < 0$.

It thus follows from above $f'(0) = 0$ and that $f'(x)$ is continuous.

Likewise $|f''(x)| < 4$ for $x > 0$ and also that

$\lim_{x\to 0} f''(x) = 4.$

Hence again implies that $f''(x)$ is continuous and $f''(0) = 4$.

On n −dimensional space let $f(x) = (f_1(x), \dots, f_n(x))$,

$M_0 = \sup|f(x)|$,

$M_1 = \sup|f'(x)|$,

and

$M_2 = \sup|f''(x)|$.

Just as in the numerical case,

there is nothing to prove if $M_2 = 0$ or $M_0 = +\infty$ or $M_2 = +\infty$.

And so we assume $0 < M_0 < +\infty$ and $0 < M_2 < \infty$.

Let a be any positive number less than M_1

Let x_0 be such that $|f'(x_0)| > a$

and let

$$u = \frac{1}{|f'(x_0)|} f'(x_0).$$

Consider the real-valued function $\varphi(x) = u.f(x)$.

Let N_0, N_1 and N_2 be the supremum of $|\varphi(x)|$, $|\varphi'(x)|$ and $|\varphi''(x)|$ respectively.

By the Schwarz inequality we have

(since $|u| = 1$) $N_0 \leq M_0$ and $N_2 \leq M_2$

While,

$N_1 \geq \varphi(x_0) = |f'(x_0)| > a$.

We therefore have

$a^2 \leq 4N_0N_2 \leq 4M_0M_2$.

Since a was any positive number less than M_1, we have

$M_1^2 \leq 4M_0M_2$

i.e., the result holds for vector-valued functions.

Equality can hold on any R^n, as we see by taking

$f(x) = (f(x), 0, \dots, 0)$

or

$f(x) = (fx), f(x), \dots, f(x))$,

where $f(x)$ is a real-valued function for which equality holds.

10.2.7 Example . Suppose f is twice-differentiable on $(0,\infty)$, f'' is bounded on $(0,\infty)$, and $f(x) \to 0$ as $x \to \infty$. Prove that $f'(x) \to 0$ as $x \to \infty$.

Solution: We shall prove an even stronger statement.

If $f(x) \to L$ as $x \to \infty$ and $f'(x)$ is uniformly continuous on $(0,\infty)$,

then $f'(x) \to 0$ as $x \to \infty$.

For, if not, let $x_n \to \infty$ be a sequence such that $f(x_n) \geq \epsilon > 0$ for all n.

(We can assume that $f(x_n)$ is a positive by replacing f with $-f$ if necessary.)

Let δ be such that

$$|f'(x) - f'(y)| < \frac{\epsilon}{2} \text{ if}$$

$$|x - y| < \delta.$$

We then have

$$f'(y) > \frac{\epsilon}{2}$$

if $|y - x_n| < \delta$, and so

$$|f(x_n + \delta) - f(x_n - \delta)| < 2\delta \cdot \frac{\epsilon}{2}$$

$$= \delta\epsilon.$$

But, since $\delta\epsilon > 0$, there exists X such that

$$|f(x) - L| < \frac{1}{2}\delta\epsilon$$

for all $x > X$.

Hence, for all large n we have

$$|f(x_n + \delta) - f(x_n - \delta)|$$

$$\leq |f(x_n + \delta) \quad L| + |L \quad f(x_n - \delta)|$$

$$< \delta\epsilon,$$

and we have reached a contradiction.

The problem follows from this result, since if f''' is bounded,

say $|f'''(x)| \leq M$, then $|f'(x) - f'(y)| \leq M|x - y|$, and

f' is certainly uniformly continuous.

10.2.8 Example . Formulate and prove an inequality which follows from Taylor's theorem and which remains valid for vector- valued functions.

Solution. There is a variety of possibilities, of which we choose just one:

Suppose $f(x)$ ahs continuous derivatives up to order n on $[a,b]$.

Then there exists $c \in (a,b)$ such that

$$|f(b) - P(b)| \leq \left|\frac{f^n(c)}{n!}\right|(b-a)^n$$

To prove this assertion true for a vector-valued function f, we merely observe that it holds for each scalar-valued function $u.f$ if u is any fixed vector of length 1.
It is obviously true if $|f(b) - P(b)| = 0$, and in all other cases it follows by taking

$$u = \frac{1}{|f(b) - P(b)|}(f(b) - P(b)).$$

10.3 SUMMARY:

This lesson introduces the concept of differentiation of vector valued functions, exploring their definitions, theorems, and properties. Learners will understand how to compute derivatives of these functions and apply differentiation to solve problems in Physics, Engineering, and Mathematics. This Lesson highlights Differentiation of vector valued functions, definitions and theorems, Properties of vector valued functions, Examples and practice problems with solutions.

10.4 TECHNICAL TERMS:

- Complex function
- Real Part
- Imaginary Part
- Differentiable function
- Continuous function
- Vector Valued function
- Scalar function
- Magnitude
- Least Upper Bounds
- Bounded
- Linear Function
- Bounded Linear Function
- n-dimensional space

10.5 SELF ASSESSMENT QUESTIONS:

1. Suppose that f is a continuous mapping of $[a,b]$ into $\mathbb{R}^k$ and f is differentiable in (a,b). Then there exists $x \in (a,b)$ such that $|f(a) - f(b)| \leq (b-a)|f'(x)|$.

2. Suppose $a \in R^1$, f is a twice-differentiable real function on (a,∞), and M_0, M_1, M_2 are the least upper bounds of $|f(x)|$, $|f'(x)|$, $|f''(x)|$ respectively, on (a,∞). Prove that $M_1^2 \leq 4M_0M_2$.

3. Suppose f is twice-differentiable on $(0,\infty)$, f'' is bounded on $(0,\infty)$, and $f(x) \to 0$ as $x \to \infty$. Prove that $f'(x) \to 0$ as $x \to \infty$.

4. Suppose f is defined in a neighbourhood of x, and suppose $f''(x)$ exists. Show that

$$\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2}=f''(x).$$

10.6 SUGGESTED READINGS:

1. Principles of Mathematical Analysis, Third Edition, Mc Graw-Hill International Editions Walter Rudin.

2. Mathematical Analysis by Tom M. Apostol, Narosa Publishing House, 2nd Edition, 1985.

Dr. V. Amarendra Babu.

LESSON-11

THE RIEMANN-STIELTJES INTEGRAL THE DEFINITION AND EXISTENCE OF THE INTEGRAL

OBJECTIVES:

The objective of the lesson is to help the learners to understand

- To understand the Definition and properties of Riemann-stieltjes integral.
- To compute the Riemann-stieltjes integral for various functions.

STRUCTURE:

11.0 INTRODUCTION:

In this lesson, the Riemann integral of a bounded real valued function is defined. A necessary and sufficient condition that a function to be Riemann integrable is proved. It is also proved that every continuous function defined on a closed interval $[a,b]$ is integrable over $[a,b]$. Further it is proved that if f is monotonic on $[a,b]$and if α is monotonically increasing and continues on $[a,b]$ then $f \in R(\alpha)$.

11.1 THE DEFINITION AND EXTSTENCE OF THE INTEGRAL:

11.1.1 Definition: Let $[a.b]$ be an interval. By a partition p. of $[a.b]$ we mean a finite set P of points $x_0, x_1, \ldots\ldots, x_n$ such that
$a = x_0 < x_1 < \cdots .. < x_n = b$.
Put $\Delta x_i = x_i - x_{i-1}$ $1 \le i \le n$. Clearly, Δx_i is the length of the sub interval $[x_{i-1}, x_i]$.

11.1.2 Definition: Let f be a bounded real valued function defined on $[a,b]$
Corresponding to each partition $P = \{x_0, x_1, x_2, \ldots\ldots, x_{n-1}, x\}$ of $[a,b]$,
We put $M_i = Sup\{f(x)/x \in [x_{i-1}, x_i]\}$ and
$m_i = Inf\left\{\frac{f(x)}{x} \in [x_{i-1}, x_i]\right\}$ for $1 \le i \le n$.
$U(p,f) = \sum_{i=1}^{n} M_i \Delta x_i$; $L(p,f) = \sum_{i=1}^{n} m_i \Delta x_i$
Put $\int_a^b f\,dx = \inf U(p,f) \ldots\ldots\ldots.(1)$ and
$\int_a^b f\,dx = \text{Sup}\, L(p,f) \ldots\ldots\ldots..(2)$
where the *Inf* and the *Sup* are taken over all partitions P of $[a,b]$
$\int_a^{\overline{b}} f\,dx$ is called the upper Riemann integral of f and

$\underline{\int_a^b} f\,dx$ is called the lower Riemann integral of f over $[a,b]$.

If $\underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx$ then we say that f is Riemann integrable over $[a,b]$. And we denote the set of all Riemann integrable functions by R and we denote the common value of (1) and (2) by $\int_a^b f\,dx$ or $\int_b^a f\,dx$.

11.1.3 Theorem: The upper and lower Riemann integrals always exist for every bounded function.

Proof: Let f be a bounded real valued function defined on $[a,b]$. Then there exist two numbers m and M such that $m \le f(x) \le M$ for all $x \in [a,b]$.

Let $P = \{x_0, x_1, x_2, \dots, x_n\}$ be any partition of $[a,b]$

Put $M_i = Sup\{f(x) | x_{i-1} \le x \le x_i\}$ and

$m_i = Inf\{f(x) | x_{i-1} \le x \le x_i\}$ for $1 \le i \le n$.

Then $m \le m_i \le M_i \le M$ for $1 \le i \le n$.

This implies $\sum_{i=1}^{n} m\Delta\alpha_i \le \sum_{i=1}^{n} m_i\Delta\alpha_i \le \sum_{i=1}^{n} M_i\Delta\alpha_i \le \sum_{i=1}^{n} M\Delta\alpha_i$

and hence $m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$

This shows that $\{L(P,f) | P$ is a partition of $[a,b]\}$ and

$\{U(P,f) | P$ is a partition of $[a,b]\}$ are bounded sets.

Therefore $Sup\{L(P,f) | P$ is a partition of $[a,b]\}$ and

$Inf\{U(P,f) | P$ is a partition of $[a,b]\}$ exist. That is $\underline{\int_a^b} f\,dx$ and $\overline{\int_a^b} f\,dx$ exist.

Thus the lower and upper Riemann integrals of a bounded function always exist.

11.1.4 Definition: Let f be a bounded real valued function defined on $[a,b]$ and

let α be a monotonically increasing function on $[a,b]$ (Then α is bounded on

$[a,b]$. For each partition $P = \{x_0, x_1, x_2, \dots, x_n\}$ of $[a,b]$ and we write

$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$ Since α is monotonically increasing on $[a,b]$, $\Delta\alpha_i \ge 0$

for $1 \le i \le n$.

Define $M_i = Sup\{f(x) | x \in [x_{i-1}, x_i]\}$ and

$m_i = Inf\{f(x) | x \in [x_{i-1}, x_i]\}$ for $1 \le i \le n$

$U(P,f,\alpha) = \sum_{i=1}^{n} M_i\Delta\alpha_i$ and $L(P,f,\alpha) = \sum_{i=1}^{n} m_i\Delta\alpha_i$

The sums $U(P,f,\alpha)$ and $L(P,f,\alpha)$ are respectively called the upper and lower

Riemann Stieltjes sums of f with respect to α corresponding to the partition P.

$\overline{\int_a^b} f\, d\alpha = Inf\{U(P,f,\alpha) | P$ is a partition of $[a,b]\}$ and

$\underline{\int_a^b} f\, d\alpha = Sup\{L(P,f,\alpha) | P$ is a partition of $[a,b]\}$

$\overline{\int_a^b} f\, d\alpha$ is called the upper Riemann-Stieltjes integral of f with respect to α over $[a,b]$ and $\underline{\int_a^b} f\, d\alpha$ is called the Lower Riemann-Stieltjes integral of f with respect to α over $[a,b]$.

If $\overline{\int_a^b} f\, d\alpha - \underline{\int_a^b} f d\alpha$, we denote the common value by $\int_a^b f d\alpha$, or by $\int_a^b f(x) d\alpha(x)$, is called Riemann - Stieltjes integral of f with respect to α over $[a,b]$.

If $\int_a^b f d\alpha$ exists, that is $\overline{\int_a^b} f\, d\alpha - \underline{\int_a^b} f d\alpha$, we say that f is integrable with respect to α in the Riemann sense. We denote the set of all Riemenn Stieltjes integrable functions with respect to α note that, by taking $\alpha(x) = x$ for all $x \in [a,b]$,the Riemann integral is seen to be a special case of the Riemann Stieltjes integral.

11.1.5 Definition: Let P be a partition of $[a,b]$. A partition P^* of $[a,b]$. is called a refinement of P if P^* contains P (i.e., if every point of P is a point of P^*).

Given two partitions P_1 and P_2 of $[a,b]$. We say that $P^*$is their common refinement if $P^* = P_1 \cup P_2$.

11.1.6 Theorem: If P^* is a refinement of P, then $L(P,f,\alpha) \leq L(P^*,f,\alpha)$ and $U(P^*,f,\alpha) \leq U(P,f,\alpha)$.

Proof: Let $P = \{x_0, x_1, x_2, \ldots.., x_n\}$ be a partition of $[a,b]$ and P^* is a refinement of P.

First suppose that P^*contains just one point more than P

Let this extra point be x^* and suppose $x_{i-1} \le x^* \le x_i$ for some i such that

$1 \le i \le n.$

Then $P = \{x_0, x_1, \dots, x_{i-1}, x^*, x_i \dots, x_n\}$

Write $m_i = Inf\{f(x) | x \in [x_{i-1,} x_i]\}$

$W_1 = Inf\{f(x) | x \in x^*, x_i]\}$ and $W_2 = Inf\{f(x) | x \in x^*, x_i]\}$

Then clearly $W_1 \ge m_i$ and $W_1 \ge m_i$

Consider
$L(P^*, f, \alpha) - L(P, f, \alpha) = m\Delta\alpha_1 + m\Delta\alpha_2 + \cdots \dots m\Delta\alpha_{i-1} + W_1[\alpha(x^*) - \alpha(x_{i-1})] + W_2[\alpha(x_i) - \alpha(x^*)] + \cdots \dots + m_n\Delta\alpha_n - \sum_{i=1}^{n} m_i\Delta\alpha_i$

That is $L(P^*, f, \alpha) - L(P, f, \alpha) \ge 0$ and hence $L(P, f, \alpha) \le L(P^*, f, \alpha)$.

If P^* have $L(P, f, \alpha) \le L(P^*, f, \alpha)$ contains k points more than P, we repeat this reasoning k times and hence
Similarly we can show that $U(P^*, f, \alpha) \le U(P, f, \alpha)$.

11.1.7 Theorem : $\underline{\int_a^b} f\, d\alpha = \overline{\int_a^b} f\, d\alpha$

Proof: For any partition P of $[a, b]$, $L(P, f, \alpha) \le U(P, f, \alpha)$

Let P^* be the common refinement of two partitions P_1 and P_2 of $[a, b]$.
By theorem 9.1.6, $L(P_1, f, \alpha) \le L(P^*, f, \alpha) \le U(P^*, f, \alpha) \le U(P, f, \alpha)$

Then $L(P_1, f, \alpha) \le U(P_2, f, \alpha) \dots \dots \dots (1)$

If P_2 is fixed and the Supremum is taken over all P_1 in (1), we have

$\overline{\int_a^b} f\, d\alpha \le U(P_2, f, \alpha) \dots \dots \dots (2)$

If the Infimum is taken over all P_2 in (1), we have

$\underline{\int_a^b} f\, d\alpha \le \overline{\int_a^b} f\, d\alpha.$

11.1.8 Theorem: $f \in R(\alpha)$ on $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$.

Proof: Assume that for each $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$.

Let $\varepsilon > 0$. Then there exists a partition P of $[a, b]$ such that

$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon \ldots\ldots\ldots(1)$

By Theorem 11.1.7, $L(P, f, \alpha) \leq \underline{\int_a^b} f\, d\alpha \leq \overline{\int_a^b} f\, d\alpha \leq U(P, f, \alpha)$

Then $0 \leq \underline{\int_a^b} f\, d\alpha - \overline{\int_a^b} f\, d\alpha < \varepsilon$

Since $\varepsilon > 0$ is arbitrary, we have $\underline{\int_a^b} f\, d\alpha = \int_a^b f\, d\alpha$

Therefore $f \in R(\alpha)$. Conversely assume that $f \in R(\alpha)$.

Then $\underline{\int_a^b} f\, d\alpha = \overline{\int_a^b} f\, d\alpha = \int_a^b f d\alpha$.

Let $\varepsilon > 0$. Then $\int_a^b f d\alpha + \frac{\varepsilon}{2}$, is not a lower bound of the set
$\{U(P, f, \alpha) | P$ is a partition of $[a, b]\}$. Then there exists a partition P_1, of $[a, b]$

such that $U(P, f, \alpha) < \int_a^b f d\alpha + \frac{\varepsilon}{2} \ldots\ldots\ldots(2)$

Now $\int_a^b f d\alpha - \frac{\varepsilon}{2}$ is not an upper bound of the set
$\{L(P, f, \alpha) | P$ is a partition of $[a, b]\}$. Then there exists a partition P_2, of $[a, b]$

such that $\int_a^b f d\alpha - \frac{\varepsilon}{2} < L(P, f, \alpha)$

This implies that $\int_a^b f d\alpha < L(P, f, \alpha) + \frac{\varepsilon}{2} \ldots\ldots\ldots.(3)$

Let P be the common refinement of two partitions P_1 and P_2 .

By Theorem 11.1.6, and by (2) and (3) we have

$U(P_1, f, \alpha) \leq U(P_2, f, \alpha) < \int_a^b f d\alpha + \frac{\varepsilon}{2} < L(P_2, f, \alpha)\varepsilon \leq L(P, f, \alpha) + \varepsilon$

This implies that $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$

Thus for given $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that

$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$.

11.1.9 Theorem: If $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$ for some partition P of $[a,b]$ and for some $\varepsilon > 0$, then $U(P^*,f,\alpha) - L(P^*,f,\alpha) < \varepsilon$ for any refinement P^* of P.

Proof: Suppose P for a partition of $[a,b]$ such that $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$ for some $\varepsilon > 0$. Let P^* be arbitrary points of P.

Then by Theorem 11.1.6.

$L(P,f,\alpha) \le L(P^*,f,\alpha) \le U(P^*,f,\alpha) \le U(P,f,\alpha)$

This implies that $U(P^*,f,\alpha) - L(P^*,f,\alpha) < \varepsilon$.

11.1.10 Theorem : If $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$ for some partition P of $[a,b]$ and for some $\varepsilon > 0$. Let s_i, t_i be arbitrary points in $[x_{i-1}, x_i]$ for $1 \le i \le n$.

Proof: Let $M_i = Sup\{f(x) | x \in [x_{i-1}, x_i]\}$ and
$m_i = Inf\{f(x) | x \in [x_{i-1}, x_i]\}$ for $1 \le i \le n$.

Then $m_i < f(s_i) < M_i$ and $m_j \le f(s_j) \le M_j$. This implies that $f(s_j) - f(t_j) \in [m_i, M_i]$ for $1 \le i \le n$.

This implies that $|f(s_j) - f(t_j)| < (m_i - M_j)$ for $1 \le i \le n$

Consider $|f(s_j) - f(t_j)| \Delta\alpha_i \le \sum_{i=1}^{n} (M_i - m_i) \Delta\alpha_i$

$\sum_{i-1}^{n} M_i \Delta\alpha_i - \sum_{i-1}^{n} m_i \Delta\alpha_i = U(P,f,\alpha) - L(P,f,\alpha)$

Therefore $\sum_{i=1}^{n} \left| f(s_i) - \int_a^b f(t_i) \right| \Delta\alpha_i < \varepsilon$

11.1.11 Theorem: If $f \in R(\alpha)$ and $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$ for a partition P of $[a,b]$ and for some $\varepsilon > 0$ if t_i is an arbitrary point in $[x_{i-1}, x_i]$ for $1 \le i \le n$ then

$$\left| \sum_{i=1}^{n} \int_a^b f(t_i) \Delta\alpha_i - \int_a^b f d\alpha \right| < \varepsilon$$

Proof: Suppose $f \in R(\alpha)$

Assume $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$ for a partition P of $[a,b]$ and for some

$\varepsilon > 0$ if t_i is an arbitrary point in $[x_{i-1}, x_i]$ for $1 \le i \le n$.

Write $M_i = Sup\{f(x) | x \in [x_{i-1}, x_i]\}$ and
$m_i = Inf\{f(x) | x \in [x_{i-1}, x_i]\}$ for $1 \le i \le n$

Now $m_i \le f(t_i) \le M_i$ for $1 \le i \le n$

Then $\sum_{i=1}^{n} m_i \Delta\alpha_i \le \sum_{i=1}^{n} f(t_i)\Delta\alpha_i \le \sum_{i=1}^{n} M_i \Delta\alpha_i$

$L(P,f,\alpha) \le \sum_{i=1}^{n} f(t_i)\Delta\alpha_i \le U(P,f,\alpha) \ldots\ldots\ldots (1)$

Since $f \in R(\alpha)$, we have $\int_a^b f d\alpha = \underline{\int_a^b} f\, d\alpha = \overline{\int_a^b} f\, d\alpha$

This implies that $L(P,f,\alpha) \le \int_a^b f d\alpha \le U(P,f,\alpha) \ldots\ldots\ldots (2)$

From (1) and (2), we have

$\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f d\alpha \le U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$ (By assumption) and

$\int_a^b f d\alpha - \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f d\alpha < \varepsilon$ and hence

$\left|\sum_{i=1}^{n} \int_a^b f(t_i)\Delta\alpha_i - \int_a^b f d\alpha\right| < \varepsilon.$

11.1.12 Theorem : If f is continuous on $[a,b]$ then $f \in R(\alpha)$on $[a,b]$.

Proof: Suppose f is continuous on $[a,b]$. Let $\varepsilon > 0$,

Since $a \le b$ and α is monotonically increasing on $[a,b]$. We have $\alpha(a) \le \alpha(b)$

This implies $\alpha(a) - \alpha(b) \ge 0$

Put $\eta_0 = \frac{\varepsilon}{\alpha(b)-\alpha(a)+1}$, then $\eta_0 > 0$.

Since f is continuous on $[a,b]$ and since $[a,b]$ is compact, by known theorem, f is

uniformly continuous $[a,b]$. Then there exists $\delta > 0$ such that
$|f(x) - f(t)| < \eta_0 \ldots\ldots\ldots (1)$, whenever $x, t \in [a,b]$and $|x - t| < \delta$.

Since $\delta > 0$, by Archimedian principle, there exists a positive integer n such that $n\delta > b \quad a$.

Write $x_i = a + \frac{t(b-a)}{n}$, $0 \leq i \leq n$.

Then P is a partition of $[a, b]$ such that $\Delta x_i - x_i - x_{i-1}$ for $1 \leq i \leq n$

$x, t \in [x_{i-1}, x_i]$, we have $|x - t| \leq \Delta x_i < \delta$

Then by (1), $|f(x) - f(t)| < \eta_0 \ldots\ldots\ldots(2)$

Write $M_i = Sup\{f(x) | x \in [x_{i\ \ 1}, x_i]\}$ and

$m_i = Inf\{f(x) | x \in [x_{i-1}, x_i]\}$ for $1 \leq i \leq n$.

Since f is continuous on $[a, b]$, Since f is also continuous on $[x_{i-1}, x_i]$ Then by Theorem there exists $p_i, q_i \in (x_{i-1}, x_i)$ such that $f(p) = m_i$ and $f(q) = M_i$ for $1 \leq i \leq n$.

Since $p_i, q_i \in [x_{i-1}, x_i]$, by (2), we have $|f(p) - f(q)| < \eta_0 \ldots\ldots\ldots(3)$

Consider $|M_i \quad m_i| = |f(p_i) \quad f(q_i)| < \eta_0$ for $1 \leq i \leq n$ By (3)

$= M_i - m_i \leq \eta_0$ for $1 \leq i \leq n \ldots\ldots\ldots(4)$

Consider $U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i - \sum_{i=1}^{n} m_i \Delta\alpha_i$

$$= \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i \leq \sum_{i=1}^{n} \eta_0 \Delta\alpha_i = \eta_0 \sum_{i=1}^{n} \Delta\alpha_i$$

$$= \eta_0\big(\alpha(b) - \alpha(a)\big) = \frac{\varepsilon\big(\alpha(b) - \alpha(a)\big)}{\big(\alpha(b) - \alpha(a)\big) + 1} < \varepsilon$$

So for given $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$. Then by Theorem 11.1.8, $f \in R(\alpha)$.

Thus every continuous function on $[a, b]$ is Riemann Stieltjes integrable over $[a, b]$.

11.1.13 Theorem: If f is monotonic on $[a, b]$ and if α is monotonically increasing and continuous on $[a, b]$, then $f \in R(\alpha)$.

Proof: Suppose f is monotonic on $[a, b]$ and α is monotonically increasing and continuous on $[a, b]$.

First we show that to each positive integer n there exists a partition

$P = \{x_0, x_1, x_2, \ldots\ldots, x_n\}$ of $[a, b]$ such that $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) = \frac{\alpha(b)-\alpha(a)}{n}$ for $1 \le i \le n$.

Let n be a positive integer.

Put $\delta = \frac{\alpha(b)-\alpha(a)}{n}$

Write $C_i = \alpha(a) - i\delta$ for $1 < i < n$.

Then $C_1 = \alpha(a) + 1\delta; C_2 = \alpha(a) + 2\delta$ and

$C_n = \alpha(a) + n\delta = \alpha(a) + \alpha(b) - \alpha(a) = \alpha(b)$

Now $\alpha(a) < C_1 < C_2 < \cdots \ldots\ldots < C_n = \alpha(b)$

Since α and continuous on $[a, b]$ and $\alpha(a) < C_1 < \alpha(b)$

by Theorem 11.1.18, there exists $x_i \in (a, b)$ such that $\alpha(x_i) = C_1$

Now $C_1 = \alpha(x_i) < C_2 < \alpha(b)$ again by Theorem 11.1.18, there exists $x_2 \in (a, b)$ such that $\alpha(x_2) = C_2$

Continuing in this way for $i = 3, 4, \ldots\ldots, n-1$, we have $x_3, x_4, \ldots\ldots, x_{n-1}$ such that $a < x_1 < x_2 < \cdots \ldots < x_{n-1} < b$ and $\alpha(x_i) = C_i$ for $1 \le i \le n$.

Put $x_0 - a$ amd $x_n - b$. Then $P - \{x_0, x_1, x_2, \ldots\ldots, x_n\}$ is a partition of $[a, b]$ and

$\Delta\alpha_i = \alpha(x_i) + \alpha(x_{i-1}) = C_i - C_{i-1} = \delta = \frac{\alpha(b)-\alpha(a)}{n}$

Therefore $\Delta\alpha_i = \frac{\alpha(b)-\alpha(a)}{n}$ for $1 \le i \le n$

So, for each positive integer n, we have a partition $P = \{x_0, x_1, x_2, \ldots\ldots, x_n\}$ of $[a, b]$ such that $\Delta\alpha_i = \frac{\alpha(b)-\alpha(a)}{n}$ for $1 \le i \le n \ldots\ldots\ldots\ldots (1)$

Let $\varepsilon > 0$

Since f is monotonic on $[a, b]$. We have either f is monotonically increasing or monotonically decreasing.

Case (i): Suppose f is monotonically increasing. Then $f(a) \le f(b)$.

Since $\varepsilon > 0$, by Archimedian principle, there exists a positive integer n such that $n\varepsilon > (\alpha(b) - \alpha(a))(f(b) - f(a))$

This implies $\Delta\alpha_i = \frac{\alpha(b)-\alpha(a)}{n}(f(b)-f(a)) < \varepsilon \ldots\ldots\ldots (2)$

For this positive integer n, by(1) we have a partition $P = \{x_0, x_1, x_2, \ldots\ldots, x_n\}$ of $[a, b]$ such that $\Delta\alpha_i = \frac{\alpha(b)-\alpha(a)}{n}$ for $1 \leq i \leq n$.

Put $M_i = Sup\{f(x) | x \in [x_{i-1}, x_i]\}$ and
$m_i = Inf\{f(x) | x \in [x_{i-1}, x_i]\}$ for $1 \leq i \leq n$

Since f is monotonically increasing we have $m_i = f(x_{i-1})$ and $M_i = f(x_i)$

Consider $U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i - \sum_{i=1}^{n} m_i \Delta\alpha_i$

$= \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i = \sum_{i=1}^{n} \left(\frac{\alpha(b)-\alpha(a)}{n}\right)(f(x_i) + f(x_{i-1}))$

$= \left(\frac{\alpha(b)-\alpha(a)}{n}\right) \sum_{i=1}^{n} (f(x_i) + f(x_{i-1}))$

$= \left(\frac{\alpha(b)-\alpha(a)}{n}\right)(f(a) + f(b)) < \varepsilon$ by(2)

Therefore $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$

Case (ii): Suppose f is monotonically decreasing. Then $f(b) \leq f(a)$.

Since $\varepsilon > 0$, by Archimedian principle, there exists a positive integer n such that
$\left(\frac{\alpha(b)-\alpha(a)}{n}\right)(f(a) - f(b)) < \varepsilon \ldots\ldots\ldots\ldots (3)$

For this positive integer n, by(1) we have a partition $P = \{x_0, x_1, x_2, \ldots\ldots, x_n\}$ of $[a, b]$ such that $\Delta\alpha_i = \frac{\alpha(b)-\alpha(a)}{n}$ for $1 \leq i \leq n$.

Since f is monotonically decreasing we have $m_i = f(x_i)$ and $M_i = f(x_{i-1})$

Consider $U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i - \sum_{i=1}^{n} m_i \Delta\alpha_i$

$= \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i = \sum_{i=1}^{n} \left(\frac{\alpha(b)-\alpha(a)}{n}\right)(f(x_{i-1}) - f(x_i))$

$= \left(\frac{\alpha(b)-\alpha(a)}{n}\right) \sum_{i=1}^{n} (f(x_i) + f(x_{i-1}))$

$= \left(\frac{\alpha(b)-\alpha(a)}{n}\right)(f(a) + f(b)) < \varepsilon$ by(3)

Thus in any case, for $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that
$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$

Then by Theorem 11.1.8, $f \in R(\alpha)$.

11.1.14 Theorem: Suppose f is bounded on $[a, b]$. f has finitely many points, of discontinuity on $[a, b]$ and α is continuous at every point at which f is discontinuous. Then show that $f \in R(\alpha)$.

Proof: Suppose f is bounded on $[a, b]$ and f has only finitely many points of discontinuity on $[a.b]$ and α is continuous at every point at which f is discontinuous.
Let $\varepsilon > 0$. Put $M = Sup\{|f(x)|| x \in [a, b]\}$

Let E be the set of points at which f is discontinuous. Then E is finite.

So let $E = \{c_1, c_2, \dots \dots \dots, c_n\}$ and assume that $c_1 < c_2 < \cdots \dots \dots < c_n$

Write $\frac{\varepsilon}{\alpha(b)-\alpha(a)+4KM+1}$ then $\varepsilon > 0$

Since α is continuous at c, there exists $\delta_j > 0$ such that $|\alpha(c_j) - \alpha(c)| < \varepsilon_1$ whenever $|c_j - c| < \delta_j$ for $j = 1, 2, \dots \dots, k$ (1)

Take $\delta_0 < min\{\delta_j, c_{j+1}, -c_j, \ 1 \le j \le k\}$

Now choose u_j and v_j such that $c_j - \frac{\delta_0}{2} < u_j < c_j < v_j + \frac{\delta_0}{2}$ for $1 \le j \le n$

Now we will show that $[u_j v_j]$'s are disjoint intervals.

For this, it is enough if we show that $v_j < u_j + 1$

Now consider $c_{j+1} - c > \delta_0$. And hence $v_j < u_{j+1}$

This implies $v_j < c_j + \frac{\delta_0}{2} < c_{j+1} - \frac{\delta_0}{2}$ and hence $v_j < u_{j+1}$

This shows that $[u_j v_j]$'s are disjoint.

Since $|c_j - v_j| < \delta_j$ and $|c_j - u_j| < \delta_j$ by (1), we have

$|\alpha(c_j) - \alpha(u_j)| < \varepsilon_1$ and $|\alpha(c_j) - \alpha(v_j)| < \varepsilon_1$ for $1 \le j \le k$

This implies that $|\alpha(v_j) - \alpha(u_j)| < 2\varepsilon_1$ for $1 \le j \le k$

Consider $\sum_{i-1}^{k}(\alpha(v_j) - \alpha(u_j)) < \sum_{i-1}^{k} 2\varepsilon_1 = 2\varepsilon_1$.

So, $\{[u_j, v_j] | 1 \le j \le k\}$ is a finite class of disjoint intervals such that $[u_j, v_j] \subseteq [a, b]$ and this class covers E and the sum of the corresponding

differences $\alpha(v_j) - \alpha(u_j)$ is less than $2k\varepsilon_1$.

Also it is clear that every point of $E \cap [a, b]$ lies in the interior of some $[u_j, v_j]$.

Write $K = [a, b] \setminus \cup_{i=1}^{k}(u_j, v_j)$

Then $K = [a, u_1] \cup [v_1, u_2] \cup [v_2, u_3] \cup \dots \dots \cup [v_k, b]$

It is clear that K is compact and f is continuous on K

By Theorem, 11.1.4, f is uniformly continuous on K. Then there exists $\delta > 0$ such that $|f(s) - f(t)| < \varepsilon_1$ whenever $|s - t| < \delta$

Now form a partition $P = \{x_0, x_1, x_2, \dots\dots, x_n\}$ of $[a, b]$ as follows: Each u_j occurs

in P. No point of any segment (u_j, v_j) occurs in P. If x_{i-1} is not one of the u_j, then

$\Delta x_i = x_i - x_{i-1} < \delta$.

Write $M_i = Sup\{f(x) | x \in [x_{i-1,} x_i]\}$ and
$m_i = Inf\{f(x) | x \in [x_{i-1,} x_i]\}$ for $1 \le i \le n$.

Assume $x_{ij} = v_i$ for $1 \le j \le k$

If $x_{i1} = v_1$ and $x_{i2} = v_2$ by the definition of P $u_1 = x_{i1} - 1$ and $u_2 = x_{i2} - 1 \dots \dots$ etc.

Therefore for any $r \in \{1,2 \dots\dots\dots n\}$. $x_r \neq x_{ij}$ implies that $x_r \neq v_j$ and $x_{r-1} \neq u_j$.

Also for any $r \in \{1,2 \dots\dots\dots n\}$, $-M \le m_r \le M_r \le M$. This implies that $M_r - m_r \le M$.

Let $r \in \{1,2 \dots\dots\dots n\} \setminus \{i_1, i_2, \dots \dots\dots, i_k\}$. Then $x_{r-j}, x_r \in K$ and $|x_r - x_{r-1}| < \delta$ by the definition of P .

Since f is continuous on $[x_{r-1}, x_r]$ by Theorem 11.1.5 there exist $s_r, t_r \in [x_{r-1}, x_r]$ such that $f(s_r) = M$, and $f(t_r) = m$. Consider $|s_r - t_r| \leq |x_r - x_{r-1}| < \delta$. This implies that $|f(s_r) - f(t_r)| < \varepsilon_1$.
Consequent $M_r - m_r = |M_r - m_r| < \varepsilon_1$.

$$\Delta\alpha_{i_1} = \alpha(x_{i_1}) - \alpha(x_{i_1 - 1}) = \alpha(v_1) - \alpha(u_1) < 2\varepsilon_1$$

$$\Delta\alpha_{i_2} = \alpha(x_{i_2}) - \alpha(x_{i_2 - 1}) = \alpha(v_2) - \alpha(u_2) < 2\varepsilon_1$$

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$$\Delta\alpha_{i_k} = \alpha(v_k) - \alpha(u_k)$$

So for any $r \in \{i_1, i_2, \ldots \ldots, i_k\}, \Delta\alpha_r < 2\varepsilon_1$

Consider $U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i - \sum_{i=1}^{n} m_i \Delta\alpha_i$

$$= \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i = \sum_{i=1}^{n} (M_r - m_r)\Delta\alpha_r\, r \in \{1,2 \ldots \ldots \ldots \ldots n\}\backslash$$
$$\{i_1, i_2, \ldots \ldots, i_k\} + \sum_{i=1}^{n} (M_r - m_r)\Delta\alpha_i < \varepsilon_1 \sum_{i=1}^{n} \Delta\alpha_i K + 2M.2\varepsilon_1 r \in \{i_1, i_2, \ldots \ldots, i_k\}$$
$$= \varepsilon_1(\alpha(b) - \alpha(a)) + 4KM\varepsilon_1 < \varepsilon_1\big((\alpha(b) - \alpha(a)) + 4KM + 1\big) = \varepsilon$$

Thus for $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that
$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$

Then by Theorem 11.1.8, $f \in R(\alpha)$.

11.1.15 Note: If f and α have a common point of discontinuity, then f need not be in $R(\alpha)$.

11.1.16 Example: Define $[-1,1] \to R$, by $\alpha(x) = 0$, if $x < 0$ and if $\alpha(x) = 1$, if $x > 1$ and $\alpha(x) = \frac{1}{2}$.

Let f be a bounded function on $[-1,1]$ such that f is not continuous at 0.

Now we will show that $f \in R(\alpha)$ on $[-1,1]$.

Let $\varepsilon > 0$

Since $f \in R(\alpha)$ there exists a partition $P = \{x_0, x_1, x_2, \ldots\ldots, x_n\}$ of $[-1,1]$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \tfrac{\varepsilon}{2} \ldots \ldots \ldots (1)$$

Now, either $0 \in P$ or $0 \notin P$

Suppose $0 \notin P$. Then $x_{i-1} < 0 < x_i$ for some i such that $1 \le i \le n$

Then $\Delta\alpha_j = 0$ for $1 < j < i - 1, \Delta\alpha_j = 0$ for $i + 1 \le j \le n$

And $\Delta\alpha_i = \alpha(x_i) \quad \alpha(x_{i-1}) = 1 \quad 0 = 1.$

Write $M_j = Sup\{f(x)|x \in [x_{i-1,}x_i]\}$ and
$m_j = Inf\{f(x)|x \subset [x_{i-1,}x_i]\}$ for $i + 1 \le j \le n$

$U(P, f, \alpha) = \sum_{i=1}^{n} M_j\Delta\alpha_j = M_j$ and $L(P, f, \alpha) = \sum_{i=1}^{n} m_j\Delta\alpha_j = m_j$

By (1) we have $M_i - m_i = U(P, f, \alpha) - L(P, f, \alpha) < \frac{\varepsilon}{2}$ … …… … (2)

Choose δ such that $0 < \delta < \min\{-x_{i-1}, x_i\}$. Then $x_{i-1} < \delta < \delta < x_{i+1}$

Suppose $x \in [-1,1]$ such that $|x - 0| < \delta$. Then $-\delta < x < \delta$.

This implies that $x_{i-1} < x < x_{i+1}$

If $x_{i-1} < x < 0 = x_i$; then $m_i \le f(x) \le M_i$ and $m_i \le f(0) \le M_i$

$|f(x) - f(0)| \le M_i - m_i \le (M_i - m_i) + (M_{i+1} - m_{i+1}) < \varepsilon$ (by 5)

If $x_i = 0 < x < x_{i+1}$, then $m_{i+1} \le f(x) \le M_{i+1}$ and $m_{i+1} \le f(0) \le M_{i+1}$

This implies that $|f(x) - f(0)| \le M_{i+1} - m_{i+1} < \varepsilon$ (by 5)

Therefore f is continuous at 0, which is contradiction.

So in any case we have contradiction.

Hence $f \in R(\alpha)$ on $[-1,1]$.

11.1.17 Theorem: Suppose $f \in R(\alpha)$ on $[a, b]$, $m \le f(x) \le M$ for all $x \in [a, b]$, φ is continuous on $[m, M]$ and $h(x) = \varphi(f(x))$on $[a, b]$. Then $h \in R(\alpha)$ on $[a, b]$.

Proof: Suppose $f \in R(\alpha)$ on $[a, b]$, $m \le f(x) \le M$ for all $x \in [a, b]$, φ is

continuous on $[m, M]$ and $h(x) - \varphi(f(x))$on $[a, b]$.

Let $\varepsilon > 0$. since φ is continuous on $[m, M]$, We have φ is bounded on $[m, M]$.

Then there exists a $\delta > 0$ such that $|\varphi(s) - \varphi(t)| < \varepsilon_1$ whenever $s, t \in [m, M]$ with $|s - t| < \delta_0$.

Choose δ such that $0 < \delta < \min\{\delta_0, \varepsilon_1\}$

Then for any $s, t \in [m, M]$ with $|s - t| < \delta$, we have $|\varphi(s) - \varphi(t)| < \varepsilon_1 \dots\dots\dots (1)$

Since $f \in R(\alpha)$ there exists a partition $P = \{x_0, x_1, x_2, \dots\dots, x_n\}$ of $[a, b]$ such that

$U(P, f, \alpha) - L(P, f, \alpha) < \delta^2 \dots\dots\dots (2)$

Write $M_i = Sup\{f(x) | x \in [x_{i-1} > x_i]\}$ and $m_i = Inf\{f(x) | x \in [x_{i-1} > x_i]\}$ and $M_i^* = Sup\{h(x) | x \in [x_{i-1} > x_i]\}$ and $m_i^* = Inf\{h(x) | x \in [x_{i-1} > x_i]\}$ for $1 \leq i \leq n$. (Since h is bounded)

Put $A = \{i \in \{1,2 \dots\dots\dots\dots\dots n\} / M_i - m_i < \delta\}$ and

$B = \{i \in \{1,2 \dots\dots\dots\dots\dots n\} / M_i - m_i > \delta\}$.

Then $A \cup B = \{1,2 \dots\dots\dots\dots\dots n\}$

First we show that $|f(x) - f(y)| \leq M_i - m_i$ for all for $x, y \in [x_{i-1}, x_i]$ for $1 \leq i \leq n$.

Let $x, y \in [x_{i-1}, x_i]$. Then $m = m_i \leq f(x) \leq M_i \leq M$ and

$m = m_i \leq f(y) \leq M_i \leq M$ implies that $f(x), f(y) \in [m, M]$ and

$|f(x) - f(y)| \leq M_i - m_i$.

Next we will show that $i \in A$ and $x, y \in [x_{i-1}, x_i]$.

Then $|f(x) - f(y)| \leq M_i - m_i < \delta$ and $f(x), f(y) \in [m, M]$

This implies that $|\varphi(f(x)) - \varphi(f(y))| < \varepsilon$ By(1)

Consequently $h(x) - h(y) < \varepsilon_1 \dots\dots\dots (3)$

Consider $M_i^* - m_i^* = Sup\{h(x) | x \in [x_{i-1}, x_i]\} - Inf\{h(y) | y \in [x_{i-1} - x_i]\}$

$= Sup\{h(x) | x \in [x_{i-1}, x_i]\} + Sup\{-h(y) | x \in [x_{i-1}, x_i]\}$

$= Sup\{h(x) - h(y) | x, y \in [x_{i-1}, x_i]\} \leq \varepsilon_1$ (by(3))

So $i \in A$ implies that $M_i^* - m_i^* \leq \varepsilon_1 \ldots\ldots\ldots(4)$

Next we will show that $i \in B$ implies that $M_i^* - m_i^* \leq 2k$.

Suppose $i \in B$. For any $x, y \in [x_{i-1}, x_i]$.

Consider
$|h(x)\text{-}h(y)| = |\varphi(f(x)) - \varphi(f(y))| \leq |\varphi(f(x))| + |\varphi(f(y))| \leq k + k = 2k.$

Therefore $M_i^* - m_i^* = Sup\{h(x) - h(y)|x, y \in [x_{i-1}, x_i]\} \leq 2k \ldots\ldots\ldots(5)$

Consider $\delta \sum_{i \in B} \Delta\alpha_i = \sum_{i \in B} \delta\Delta\alpha_i \leq \sum_{i \in B}(M_i - m_i)\Delta\alpha_i$

$\leq \sum_{i=1}^{n}(M_i^* - m_i^*)\Delta\alpha_i = U(P, f, \alpha) - L(P, f, \alpha) < \delta^2$ (by(2))

This implies that $\delta \sum_{i \in B} \Delta\alpha_i < \delta^2$ and hence $\sum_{i \in B} \Delta\alpha_i < \delta \ldots\ldots\ldots(6)$

Now consider $U(P, h, \alpha) - L(P, h, \alpha) = \sum_{i=1}^{n}(M_i^* - m_i^*)\Delta\alpha_i$

$$= \sum_{i \in A}(M_i^* - m_i^*)\Delta\alpha_i + \sum_{i \in B}(M_i^* - m_i^*)\Delta\alpha_i$$

$\leq \varepsilon_1 \sum_{i \in A} \Delta\alpha_i + 2k \sum_{i \in B} \Delta\alpha_i$ (By (4) and (5))

$\leq \varepsilon_1 \sum_{i=1}^{n} \Delta\alpha_i + 2k \sum_{i \in B} \Delta\alpha_i < \varepsilon_1(\alpha(b) - \alpha(a)) + 2kf$ (By (6))

$\leq \varepsilon_1(\alpha(b) - \alpha(a)) + 2k\varepsilon_1$

$= \varepsilon_1(\alpha(b) - \alpha(a) + 2k)$

$\leq \varepsilon_1(\alpha(b) - \alpha(a) + 2k + 1) = \varepsilon$

Thus for given $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$U(P, h, \alpha) - L(P, h, \alpha) < \varepsilon$ and hence $h \in R(\alpha)$ on $[a, b]$.

11.1.18 Problem: If $f(x) = 0$ for all irrational x and $f(x) = 1$ for all rational x prove that $f \notin R(\alpha)$ on $[a, b]$ for any $a < b$.

Solution: Let a, b be real numbers such that $a < b$.

Let $f: [a, b] \to \mathbb{R}$ be the function defined by $f(x) = 0$ for all irrational x and $f(x) = 1$ for all rational x.

Let $P = \{x_0, x_1, x_2, \ldots\ldots, x_n\}$ be any partition of $[a, b]$

Write $M_i = Sup\{f(x) | x \in [x_{i-1}, x_i]\}$ and $m_i = Inf\{f(x) | x \in [x_{i-1}, x_i]\}$ for $1 \le i \le n$.

Then $M_i = 1$ and $m_i = 0$ for $1 \le i \le n$.

$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta x_i = b - a$ and $L(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta x_i = 0$

Then $\underline{\int_a^b} f\, dx = Sup\{L(P, f) | P \text{ is a partition of } [a,b]\} = 0$

And $\overline{\int_a^b} f\, dx = Inf\{U(P, f) | P \text{ is a partition of } [a,b]\} = b - a$

Therefore $\underline{\int_a^b} f\, dx \neq \overline{\int_a^b} f\, dx$ and hence $f \notin R(\alpha)$ on $[a, b]$.

11.1.19 Problem: Suppose $f \ge 0$, f is continuous on $[a, b]$ and $\int_a^b f\, dx$

prove that for all $x \in [a, b]$.

Solution: Suppose $f \ge 0$, f is continuous on $[a, b]$ and $\int_a^b f\, dx = 0$

If possible suppose that $f(c) \neq 0$ for some $c \in [a, b]$, then $f(c) > 0$. Since f is

continuous on $[a, b]$, f is continuous at c. Then there exists $\delta > 0$ such that $f(x) - f(c) < f(c) \ldots\ldots\ldots (1)$ whenever $x \in [a, b]$ with $|x - c| < \delta$.

Now we will show that $f(x) \neq 0$ for all $x \in (c - \delta, c + \delta)$. If possible suppose

that $f(x) = 0$ for some $x \in (c - \delta, c + \delta)$. Then $|x - c| < \delta$, and

by (1), $|f(x) - f(c)| < f(c)$.

Since $f(x) = 0$, we have $f(c) < f(c)$, a contradiction.

So $f(x) \neq 0$ for all $x \in (c - \delta, c + \delta)$.

Since $f(x) \ge 0$ on $[a, b]$. We have $f(x) > 0$ for all $x \in (c - \delta, c + \delta)$.

This implies $\int_{c-\delta}^{c+\delta} f(x)\, dx > 0$ and hence $\int_a^b f(x)\, dx \neq 0$ a contradiction.

So $f(x) = 0$ for all $x \in [a, b]$.

11.1.20 Problem: Suppose α increases on $[a, b]$ and $a < s < b$ and α is continuous at s, $f(s) - 1$ and $f(x) \neq 0$ if $x \neq s$. Prove that $f \in R(\alpha)$ and that $\int_a^b f\, d\alpha = 0$.

Solution : Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be any partition of $[a, b]$.

Then $x_{i-1} < s < x_i$ for some i such that $1 \leq i \leq n$.

Write $M_j = Sup\{f(x) | x \in [x_{j-1}, x_j]\}$ and
$m_j = Inf\{f(x) | x \in [x_{j-1}, x_j]\}$ for $1 \leq j \leq n$.

Then $M_j - 0$ for $1 \leq j \leq n$ and $j \neq i$ and $M_i = 1$ and $m_i = 0$ for $1 \leq j \leq n$.

Now $L(P, f, \alpha) = 0$ and $U(P, f, \alpha) = M_i \Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) \geq 0$

Therefore $\underline{\int_a^b} f\, d\alpha = Sup\{L(P, f, \alpha) | P$ is a partition of $[a, b]\} = 0$

And $\overline{\int_a^b} f\, d\alpha = Inf\{U(P, f, \alpha) | P$ is a partition of $[a, b]\} \geq 0$

Now we will show that $\overline{\int_a^b} f\, d\alpha - 0$

If possible suppose that $\overline{\int_a^b} f\, d\alpha > 0$. Choose ε such that $0 < \varepsilon < \overline{\int_a^b} f\, d\alpha$

Since α is continuous at s, there exists $\delta > 0$ such that $0 < s - \delta < s < s + \delta < b$ and $\alpha(s) - \alpha(x) < \frac{\varepsilon}{2} \ldots\ldots\ldots (1)$ whenever $|s - x| < \delta$.

Take $x_0 = a, x_1 = s \quad \frac{\delta}{2}, x_2 = s \mid \frac{\delta}{2}, x_3 = b$.

Then $P = \{x_0, x_1, x_2, \ldots, x_n\}$ is a partition of $[a, b]$ and $x_1 < s < x_2$

Clearly, $|s - x_1| = \left|s - \left(s - \frac{\delta}{2}\right)\right| = \frac{\delta}{2} < \delta$

Then by (1) $|\alpha(s) - \alpha(x_1)| < \frac{\varepsilon}{2}$

Clearly $M_1 = 0, M_2 = 1, M_3 = 0$

Consider $|s - x_2| = \left|s - \left(s - \frac{\delta}{2}\right)\right| = \frac{\delta}{2} < \delta$

Then by (1) $|\alpha(s) - \alpha(x_2)| < \frac{\varepsilon}{2}$

Consider $|\alpha(x_2)-\alpha(x_1)| \le |\alpha(x_2)-\alpha(s)| + |\alpha(s)-\alpha(x_1)| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$

Therefore $|\alpha(x_2)-\alpha(x_1)| = \varepsilon \ldots\ldots\ldots..(2)$

$U(P,f,\alpha) = \sum_{i=1}^{3} M_i \Delta\alpha_i = M_2 \Delta\alpha_i(x_2) - \alpha_i(x_1) < \varepsilon$

Thus there exists a partition P of $[a,b]$ such that $U(P,f,\alpha) < \overline{\int_a^b} f\, d\alpha$.

Which is a contradiction.

So $\overline{\int_a^b} f\, d\alpha = 0$ and hence $\int_a^b f\, d\alpha = \overline{\int_a^b} f\, d\alpha = 0$

Consequently, $f \in R(\alpha)$ on $[a,b]$ and $\int_a^b f\, d\alpha = 0$.

Short Answer Questions

1. Define the upper Riemann integral and lower Riemann integral of a bounded function f defined on $[a.b]$.

2. Show that $\underline{\int_a^b} f\, d\alpha \le \overline{\int_a^b} f\, d\alpha$

3. If $f(x) = 0$ for all irrational x and $f(x) = 1$ for all rational x prove that $f \notin R(\alpha)$ on $[a,b]$ for any $a < b$.

Model Examination Questions

1. Show that $f \in R(\alpha)$ on $[a,b]$ if and only if for every $\varepsilon > 0$ there exists a partition P of $[a,b]$ such that $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$.

2. If f is continuous on $[a,b]$ then show that $f \in R(\alpha)$ on $[a.b]$.

3. If f is monotonic on $[a,b]$ and if α is monotonically increasing and continuous on $[a,b]$, then $f \in R(\alpha)$.

4. Suppose f is bounded on $[a,b]$.f has finitely many points, of discontinuity on $[a,b]$ and α is continuous at every point at which f is discontinuous. Then show that $f \in R(\alpha)$.

5. Suppose α increases on $[a,b]$ and $a < s < b$ and α is continuous at s, $f(s) = 1$ and $f(x) \neq 0$ if $x \neq s$. Prove that $f \in R(\alpha)$ and that $\int_a^b f\, d\alpha = 0$.

Exercises

1. Define $[-1,1] \to \mathbb{R}$, by $\beta(x) = 0$,if $x < 0$ and if $\beta(x) = 1$,if $x > 0$ and Let f be a bounded function on $[-1,1]$. Show that $f \in R(\beta)$ if and only if $f(0+) = f(0)$ and that then $\int_{-1}^{1} f\, d\beta = 0$.

Answers to Short Answer Questions:

For **1** see definition **11.1.2**

For **2** see theorem **11.1.7**

For **3** see problem **11.1.18**

11.2 SUMMARY:

This lesson delves into the concept of the Riemann-Stieltjes integral, covering its definition, properties, and applications. Learners will gain a comprehensive understanding of the integral's existence and computation for various functions. This Lesson covers Introduction to the Riemann-Stieltjes integral, Definition and existence of the integral, Proofs of key theorems, Computation of the Riemann-Stieltjes integral for various functions, Practice problems with solutions and exercise problems.

11.3 TECHNICAL TERMS

- Interval
- Partition
- Supremum and Infimum
- Riemann Stieltjes integral
- Bounded function
- Monotonic
- Upper and Lower Riemann Stieltjes integral
- Refinement
- Compact
- Continuous
- Uniformly Continuous

11.4 SELF ASSESSMENT QUESTIONS

1. If f is monotonic on $[a, b]$ and if α is monotonically increasing and continuous on $[a, b]$, then $f \in R(\alpha)$.

2. Show that $f \in R(\alpha)$ on $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$.

3. If f is monotonic on $[a, b]$ and if α is monotonically increasing and continuous on $[a, b]$, then $f \in R(\alpha)$.

4. If $f(x) = 0$ for all irrational x and $f(x) = 1$ for all rational x prove that $f \notin R(\alpha)$ on $[a, b]$ for any $a < b$.

5. Show that $\underline{\int_a^b} f\, d\alpha \leq \overline{\int_a^b} f\, d\alpha$.

11.5 SUGGESTED READINGS:

1. Principles of Mathematical Analysis, Third Edition, Mc Graw-Hill International Editions Walter Rudin.

2. Mathematical Analysis by Tom M. Apostol, Narosa Publishing House, 2nd Edition, 1985.

I. V. Venkateswara Rao

LESSON-12

PROPERTIES OF RIEMANN STIELTJES INTEGRAL

OBJECTIVES:

The objective of the lesson is to help the learners to understand

- To understand the Definition and properties of Riemann-stieltjes integral.
- To compute the Riemann-stieltjes integral for various functions.

STRUCTURE:

12.0 INTRODUCTION:

In this lesson the properties of Riemann-Stieltjes integral are studied. If $\boldsymbol{R_\alpha(a,b)}$ denotes the set of all real-valued functions $\boldsymbol{f}$ defined on $[\boldsymbol{a},\boldsymbol{b}]$ such that $\boldsymbol{f \in R(\alpha)}$ on $[\boldsymbol{a},\boldsymbol{b}]$, then it is proved that $\boldsymbol{f+g}$ and $\boldsymbol{cf}$ are in $\boldsymbol{R_\alpha(a,b)}$ for any $\boldsymbol{f g \in R_\alpha(a,b)}$ and for any real number $\boldsymbol{c}$. This shows that $\boldsymbol{R_\alpha(a,b)}$ is a vector space over the field of real numbers. Further it is proved that if $\boldsymbol{a < s < b}$, $\boldsymbol{f}$ is bounded on $[\boldsymbol{a},\boldsymbol{b}]$ $\boldsymbol{f}$ is continuous at $\boldsymbol{s}$ and $\boldsymbol{\alpha(x) = I(x-s)}$, then $\int_a^b \boldsymbol{f\,dx = f(s)}$.

12.1 PROPERTIES OF INTEGRAL

12.1.1 Theorem: If $f_1 \in R(\alpha)$ and $f_2 \in R(\alpha)$ on $[a,b]$ then $f_1+f_2 \in R(\alpha)$ and

$$\int_a^b (f_1+f_2)\,d\alpha = \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha.$$

Proof: Suppose $f_1 \in R(\alpha)$ and $f_2 \in R(\alpha)$ on $[a,b]$

Put $f = f_1 + f_2$

Let $\varepsilon > 0$

Since $f_1 \in R(\alpha)$ and $f_2 \in R(\alpha)$ on $[a,b]$, by Theorem 11.1.8, there exist partitions P_1 and P_2 $[a,b]$ such that $U(P_1,f_1,\alpha) - L(P_1,f_1,\alpha) < \frac{\varepsilon}{2}$.

Let P be the common refinement of P_1 and P_2. Then by Theorem 11.1.9.

$U(P,f_1,\alpha) - L(P,f_1,\alpha) < \frac{\varepsilon}{2}$ ……………………….(1)

And $U(P,f_2,\alpha) - L(P,f_2,\alpha) < \frac{\varepsilon}{2}$ ……………….(2)

$Inf\{f(x) | x \in [x_{i-1},x_i]\}$

$\geq Inf\{f_1(x) | x \in [x_{i-1},x_i]\} + Inf\{f_2(x) | x \in [x_{i-1},x_i]\}$……………………(3)

$Sup\{f(x)|x \in [x_{i-1,}x_i]\}$
$\leq Sup\{f_1(x)|x \in [x_{i-1,}x_i]\} + Sup\{f_2(x)|x \in [x_{i-1,}x_i]\}$.......................(4)
From (3) and (4), we have
$L(P, f_1, \alpha) + L(P, f_2, \alpha) \leq L(P, f, \alpha) \leq U(P, f, \alpha) + U(P, f_1, \alpha) + U(P, f_2, \alpha)$
This implies $U(P, f, \alpha) - L(P, f, \alpha)$
$\leq U(P, f_1, \alpha) - L(P, f_1, \alpha) + U(P, f_2, \alpha) - L(P, f_2, \alpha)$
$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$ By(1) and (2).
Thus for Since $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that
$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$.
Therefore $f \in R(\alpha)$ That is $f_1 + f_2 \in R(\alpha)$.
Next we will show that $\int_a^b (f_1 + f_2)\, d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha$.
Let ε be an arbitrary positive real number.
Since $f_1 f_2 \in R(\alpha)$ on $[a, b]$, $\underline{\int_a^b} f_1\, d\alpha = \overline{\int_a^b} f_1\, d\alpha = \int_a^b f_1\, d\alpha$ and
$\underline{\int_a^b} f_2\, d\alpha = \overline{\int_a^b} f_2\, d\alpha = \int_a^b f_2\, d\alpha$.
For $i = 1, 2$ $\int_a^b f_j\, d\alpha + \frac{\varepsilon}{2}$ is not a lower bound of the set $U(P, f_j, \alpha)$ is a partition of $[a, b]$.
Then $U(P_j, f_j, \alpha) < \int_a^b f_j\, d\alpha + \frac{\varepsilon}{2}$ for some partitions P_j of $[a, b]$ for $j = 1,2, \ldots \ldots$
Let P be the common refinement of P_1 and P_2.
Then $U(P, f_j, \alpha) \leq U(P_j, f_j, \alpha) < \int_a^b f_j\, d\alpha + \frac{\varepsilon}{2}$ for $j = 1,2 \ldots \ldots$
Now
$\int_a^b f\, d\alpha \leq U(P, f, \alpha) \leq U(P, f_1, \alpha) + U(P, f, \alpha) \leq U(P_1, f_1, \alpha) + U(P_2, f_2, \alpha) <$
$\int_a^b f_1\, d\alpha + \frac{\varepsilon}{2} + \int_a^b f_2\, d\alpha + \frac{\varepsilon}{2} \ldots \ldots (5)$

This implies $\int_a^b f\, d\alpha \leq \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha + \varepsilon$
Since $\varepsilon > 0$ is arbitrary, we have $\int_a^b f\, d\alpha \leq \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha \ldots \ldots (6)$
For $j = 1, 2$ $\int_a^b f_j\, d\alpha - \frac{\varepsilon}{2}$ is not an upper bound of the set $L(P, f_j, \alpha)$ is a partition of $[a, b]$.
Then $\int_a^b f_j\, d\alpha - \frac{\varepsilon}{2} < L(P_j, f_j, \alpha)$ for some partitions P_j of $[a, b]$ for $j = 1,2, \ldots \ldots$
This implies that
$\int_a^b f_1\, d\alpha - \frac{\varepsilon}{2} +$
$\int_a^b f_2\, d\alpha - \frac{\varepsilon}{2} < L(P, f_1, \alpha) + L(P, f_2, \alpha) + L(P, f, \alpha) \leq \int_a^b f\, d\alpha \ldots \ldots (7)$
Therefore $\int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha - \varepsilon < \int_a^b f\, d\alpha$
Since $\varepsilon > 0$ is arbitrary, we have

$\int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha \le \int_a^b f\, d\alpha \ldots\ldots\ldots (8)$

From (6) and (8), $\int_a^b f\, d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha$

Then $\int_a^b (f_1 + f_2)\, d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha.$

12.1.2 Theorem: If $f \in R(\alpha)$ on $[a, b]$ and c is any constant, then $cf \in R(\alpha)$ on $[a, b]$ and $\int_a^b cf\, d\alpha = c\int_a^b f\, d\alpha$.

Proof: Suppose $f \in R(\alpha)$ on $[a, b]$ and c is any constant.

If $c = 0$, then clearly $cf \in R(\alpha)$

Let $\varepsilon > 0$. Suppose $c > 0$

Since $f \in R(\alpha)$ on $[a,b]$, there exists a partition $P = \{x_0, x_1, x_2, \ldots\ldots, x_n\}$ of $[a, b]$ such that $U(P, f, \alpha) - L(P, f, \alpha) < \frac{\varepsilon}{c}$

Write $M_i = Sup\{f(x) | x \in [x_{i-1}, x_i]\}$ and

$m_i = Inf\{f(x) | x \in [x_{i-1}, x_i]\}$ for $1 \le i \le n.$

Consider $Sup\{(cf)(x) | x \in [x_{i-1}, x_i]\} = Sup\{cf(x) | x \in [x_{i-1}, x_i]\} = cM_i$

Similarly, $Inf\{(cf)(x) | x \in [x_{i-1}, x_i]\} = Inf\{cf(x) | x \in [x_{i-1}, x_i]\} = cm_i$ for $1 \le i \le n$.

Consider $U(P, cf, \alpha) - L(P, cf, \alpha) = \sum_{i=1}^{n} cM_i\Delta\alpha_i - \sum_{i=1}^{n} cm_i\Delta\alpha_i$

$$= c\left\{\sum_{i=1}^{n} M_i\Delta\alpha_i - \sum_{i=1}^{n} m_i\Delta\alpha_i\right\}$$

$$= c\{U(P, f, \alpha) - L(P, f, \alpha)\} < c.\frac{\varepsilon}{c} = \varepsilon$$

Therefore, $U(P, cf, \alpha) - L(P, cf, \alpha) < \varepsilon$ for some partition P of $[a, b]$

And hence $cf \in R(\alpha)$

So in this case $cf \in R(\alpha)$.

Suppose $c < 0$ then $-c > 0$.

Since $f \in R(\alpha)$, there exists a partition P of $[a, b]$ such that

$U(P, f, \alpha) - L(P, f, \alpha) < \frac{\varepsilon}{-c}$

Consider $Sup\{(cf)(x) | x \in [x_{i-1}, x_i]\} = Sup\{cf(x) | x \in [x_{i-1}, x_i]\}$

$= -c\, Sup\{(-f)(x) | x \in [x_{i-1}, x_i]\} = cM_i$

Similarly $Inf\{(cf)(x) | x \in [x_{i-1}, x_i]\} = Inf\{cf(x) | x \in [x_{i-1}, x_i]\}$

$= -c\, Inf\{(-f)(x) | x \in [x_{i-1}, x_i]\} = cm_i$

This implies $U(P, cf, \alpha) = L(P, cf, \alpha)$

Similarly we can show that $L(P, cf, \alpha) = c\, U(P, cf, \alpha)$

Consider $U(P, cf, \alpha) - L(P, cf, \alpha) = c\, L(P, f, \alpha) - c\, U(P, f, \alpha)$

$$= -c\{U(P, f, \alpha) - L(P, f, \alpha)\} < -c.\frac{\varepsilon}{-c} = \varepsilon$$

Therefore $U(P, cf, \alpha) - L(P, cf, \alpha) < \varepsilon$ for some partition P of $[a, b]$

And hence $cf \in R(\alpha)$.

Thus in any case $cf \in R(\alpha)$.

Next we will show $\int_a^b cf\, d\alpha = c\int_a^b f\, d\alpha$

Since $f \in R(\alpha)$., we have $\int_a^b cf\, d\alpha - \int_{\underline{a}}^b cf\, d\alpha = \int_a^{\overline{b}} cf\, d\alpha$

If $c = 0$, then clearly $\int_a^b cf\, d\alpha = c\int_a^b f\, d\alpha$

Since $cf \in R(\alpha)$, we have $\int_a^b cf\, d\alpha = \int_{\underline{a}}^b cf\, d\alpha = \int_a^{\overline{b}} cf\, d\alpha$

Suppose $c < 0$

Then $U(P, cf, \alpha) - L(P, cf, \alpha)$ and $L(P, cf, \alpha) = c\, U(P, cf, \alpha)$ for any partition P of $[a, b]$.

Consider $\int_a^b cf\, d\alpha = \int_a^{\overline{b}} cf\, d\alpha$

$= Sup\{L(P, cf, \alpha) | P \text{ is a partition of } [a, b]\}$

$= Sup\{cU(P, f, \alpha) | P \text{ is a partition of } [a, b]\}$

$= -c, Sup\{-U(P, f, \alpha) | P \text{ is a partition of } [a, b]\}$

$= -c, Inf\{U(P, f, \alpha) | P \text{ is a partition of } [a, b]\}$

$= c, Inf\{U(P, f, \alpha) | P \text{ is a partition of } [a, b]\}$

$= c\int_a^b f\, d\alpha$

Therefore $\int_a^b cf\, d\alpha = c\int_a^b f\, d\alpha$

Similarly we can prove for $c > 0$, we have $\int_a^b cf\, d\alpha = c\int_a^b f\, d\alpha$

Thus in any case $\int_a^b cf\, d\alpha = c\int_a^b f\, d\alpha$.

12.1.3 Theorem: If $f_1, f_2 \in R(\alpha)$ on $[a, b]$ and $f_1(x) \leq f_2(x)$ on $[a, b]$ then $\int_a^b f_1\, d\alpha \leq \int_a^b f_2\, d\alpha$.

Proof: Suppose $f_1, f_2 \in R(\alpha)$ on $[a, b]$ and $f_1(x) \leq f_2(x)$ on $[a, b]$

Let $P = \{x_0, x_1, x_2, \ldots\ldots, x_n\}$ be any partition of $[a, b]$

Write $M_i - Sup\{f_1(x) | x \in [x_{i-1}, x_i]\}$ and

$N_i = sup\{f_2(x) | x \in [x_{i-1}, x_i]\}$ for $1 \leq i \leq n$.

Since $f_1(x) \leq f_2(x)$ for all $x \in [a, b]$,

we have $f_1(x) \leq f_2(x)$ for all $x \in [x_{i-1}, x_i]$ for $1 \leq i \leq n$.

Then $M_i \leq N_i$ for $1 \leq i \leq n$.

This implies that $U(P, f_1, \alpha) \leq U(P, f_2, \alpha)$

Consider $\int_a^b f_1\, d\alpha \leq \int_a^{\overline{b}} f_2\, d\alpha \leq U(P, f_1, \alpha) \leq U(P, f_2, \alpha)$

This shows that $\int_a^b f_1\, d\alpha$ is a lower bound of $\{U(P, f_2, \alpha) | P \text{ is a partition of } [a, b]\}$.

Therefore $\int_a^b f_1\, d\alpha \leq \int_a^{\overline{b}} f_2\, d\alpha = \int_a^b f_2\, d\alpha$

Thus $\int_a^b f_1\, d\alpha \leq \int_a^b f_2\, d\alpha$.

12.1.4 Theorem: If $f \in R(\alpha)$ on $[a,b]$ and $a < c < b$, then $f \in R(\alpha)$ on $[a,c]$ and on $[c,b]$ and

$$\int_a^c f\,d\alpha + \int_b^c f\,d\alpha = \int_a^b f\,d\alpha$$

Proof: Suppose $f \in R(\alpha)$ on $[a,b]$ and $a < c < b$

Let $\varepsilon > 0$

First we show that there exists a partition P of $[a,b]$ such that $c \in [a,b]$ and

$U(Q,f,\alpha) - L(Q,f,\alpha) < \varepsilon$

Since $f \in R(\alpha)$, there exists a partition $Q = \{x_0, x_1, x_2, \ldots, x_n\}$ of $[a,b]$ such that

$U(Q,f_2,\alpha) - L(Q,f_2,\alpha) < \varepsilon$

Since $c \in [a,b]$, we have either $c = x_i$ or $x_{i-1} < c < x_i$ for some i such that $1 \le i \le n$

If $c = x_i$, then $c \in Q$. So Q is a partition of $[a,b]$ such that $c \in Q$ and

$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon,$

Suppose $x_{i-1} < c < x_i$. Then $P = \{x_0, x_1, x_2, \ldots, x_{i-1}, c, x_i, \ldots, x_n\}$ is a partition of $[a,b]$ which is a refinement of Q.

Then by Theorem 11.1.6., $L(Q,f,\alpha) \le L(P,f,\alpha) \le U(P,f,\alpha) \le U(Q,f,\alpha)$

This implies that $U(P,f,\alpha) - L(P,f,\alpha) \le U(Q,f,\alpha) - L(Q,f,\alpha) < \varepsilon$

So, there exists a partition P of $[a,b]$ such that $c \in P$ and $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$.

Assume that the above partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ and $x_{i_0} = c$ for some i_0 such that $1 \le i_0 \le n$.

Write $Q_1 = \{x_0, x_1, x_2, \ldots, x_{i_0}\}$ and $Q = \{x_0, x_1, x_2, \ldots, x_n\}$

Then Q_1 is a partition of $[a,c]$ and Q_2 is a partition of $[c,b]$

f on $[a,c]$ is f_1 and f on $[c,b]$ is f_2

Write $M_i = Sup\{f(x) | x \in [x_{i-1}, x_i]\}$ and

$m_i = Inf\{f(x) | x \in [x_{i-1}, x_i]\}$ for $1 \le i \le n$.

Consider $U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i + \sum_{i=1}^{i_0} M_i \Delta\alpha_i + \sum_{i=i_0+1}^{n} M_i \Delta\alpha_i$

$= U(Q_1,f_1,\alpha) + U(Q_2,f_2,\alpha)$

Similarly $U(P,f,\alpha) = L(Q_1,f_1,\alpha) + L(Q_2,f_2,\alpha)$

Consider $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$. This implies that

$U(Q_1,f_1,\alpha) - L(Q_1,f_1,\alpha) < \varepsilon$ and $U(Q_2,f_2,\alpha) - L(Q_2,f_2,\alpha) < \varepsilon$.

That is $U(Q_1,f_1,\alpha) - L(Q_1,f_1,\alpha) < \varepsilon$ on $[a,c]$

And $U(Q_2,f_2,\alpha) - L(Q_2,f_2,\alpha) < \varepsilon$ on $[c,b]$

By Theorem 11.1.8 $f \in R(\alpha)$ on $[a,c]$ and $f \in R(\alpha)$ on $[c,b]$

Next we will show that $\int_a^c f\,d\alpha + \int_b^c f\,d\alpha = \int_a^b f\,d\alpha$.

Let $\varepsilon > 0$

Since $f \in R(\alpha)$ there exists a partition P of $[a,b]$ such that

$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon \ldots\ldots\ldots (1)$

Without loss of generality we may assume that $c \in P$ and suppose $x_{i_0} = c$ for some i_0 such that $1 \le i_0 \le n$.

Write $Q_1 = \{x_0, x_1, x_2, \ldots, x_{i_0}\}$ and $Q = \{x_0, x_1, x_2, \ldots, x_n\}$

Then Q_1 is a partition of $[a, c]$ and Q_2 is a partition of $[c, b]$ such that
$P = Q_1 \cup Q_2$

Now $\int_a^b f\,d\alpha \le U(P, f, \alpha) < L(P, f, \alpha) + \varepsilon$ (By(1))

$$= L(Q_1, f, \alpha) + L(Q_2, f, \alpha) + \varepsilon \le \int_a^c f\,d\alpha + \int_a^b f\,d\alpha + \varepsilon$$

This implies that $\int_a^b f\,d\alpha < \int_c^c f\,d\alpha + \int_c^b f\,d\alpha + \varepsilon$

Since $\varepsilon > 0$ is arbitrary, we have $\int_a^b f\,d\alpha \le \int_a^c f\,d\alpha + \int_c^b f\,d\alpha \ldots\ldots\ldots(2)$

Consider $\int_a^b f\,d\alpha \ge L(P, f, \alpha) > U(P, f, \alpha) - \varepsilon$

$$= U(Q_1, f, \alpha) + U(Q_2, f, \alpha) - \varepsilon$$

$$\ge \int_a^c f\,d\alpha + \int_a^b f\,d\alpha - \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we have $\int_a^b f\,d\alpha \ge \int_a^c f\,d\alpha + \int_c^b f\,d\alpha \ldots\ldots\ldots(3)$

From (2) and (3) $\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$

12.1.5 Theorem: $f \in R(\alpha)$ on $[a, b]$ and if $|f(x)| \le M$ on $[a, b]$ then
$\left|\int_a^b f\,d\alpha\right| \le M[\alpha(b) - \alpha(a)]$.

Proof: Suppose $f \in R(\alpha)$ on $[a, b]$ and $|f(x)| \le M$ on $[a, b]$.

Since $f \in R(\alpha)$ on $[a, b]$, we have $\int_a^b f\,d\alpha = \underline{\int_a^b} f\,d\alpha + \overline{\int_a^b} f\,d\alpha$

Let P be a partition of $[a, b]$

Write $M_i = Sup\{f(x) | x \in [x_{i-1,} x_i]\}$ and

$m_i = Inf\{f(x) | x \in [x_{i-1,} x_i]\}$ for $1 \le i \le n$.

Since $|f(x)| \le M$ on $[a, b]$, we have $-M < f(x) < M$ for all $x \in [a, b]$.

This implies that $-M \le f(x) \le M$ for all $x \in [x_{i-1,} x_i]$ for $1 \le i \le n$ and hence
$-M \le m_i \le M_i \le M$ for $1 \le i \le n$.

Consider $L(P, f, \alpha) < \int_a^b f\,d\alpha = \underline{\int_a^b} f\,d\alpha + \overline{\int_a^b} f\,d\alpha \ldots\ldots\ldots(1)$

Consider $L(P, f, \alpha) = \sum_{i=1}^n M_i \Delta\alpha_i \ge \sum_{i=1}^n (-M)\Delta\alpha_i = -M\sum_{i=1}^n \Delta\alpha_i$
$= -M[\alpha(b) - \alpha(a)]$

This implies $L(P, f, \alpha) > -M[\alpha(b) - \alpha(a)] \ldots\ldots\ldots(2)$

Consider $U(P, f, \alpha) = \sum_{i=1}^n M_i \Delta\alpha_i \le \sum_{i=1}^n (M)\Delta\alpha_i = M\sum_{i=1}^n \Delta\alpha_i$
$= M[\alpha(b) - \alpha(a)]$

This implies $U(P, f, \alpha) \le M[\alpha(b) - \alpha(a)] \ldots\ldots\ldots(3)$

From (1), (2) and (3), we have

$-M[\alpha(b) - \alpha(a)] \le \int_a^b f\,d\alpha \le M[\alpha(b) - \alpha(a)]$

And hence $\left|\int_a^b f\,d\alpha\right| \le M[\alpha(b) - \alpha(a)]$.

12.1.6 Theorem: If $f \in R(\alpha_1)$ and $f \in R(\alpha_2)$ on $[a,b]$ then $f \in R(\alpha_1+\alpha_2)$ on $[a,b]$ and

$$\int_a^b f d(\alpha_1+\alpha_2) = \int_a^b f d\,\alpha_1 + \int_a^b f d\,\alpha_2$$

Proof: Suppose $f \in R(\alpha_1)$ and $f \in R(\alpha_2)$ on $[a,b]$. Let $\varepsilon > 0$.

Since $f \in R(\alpha_1)$ on $[a,b]$ for $j = 1,2$ there exists P_j of $[a,b]$ such that

$U(P_j, f, \alpha_j) - L(P_j, f, \alpha_j) < \frac{\varepsilon}{2} \ldots\ldots\ldots (1)$

Let P be the common refinement of P_1 and P_2.

Then by Theorem 11.1.6, $L(P_j, f, \alpha_j) \le L(P_j, f, \alpha_j) \le U(P_j, f, \alpha_j) \le U(P_j, f, \alpha_j)$ for $j = 1,2$.

$U(P, f, \alpha_j) - L(P_j, f, \alpha_j) \le U(P_j, f, \alpha_j) - L(P_j, f, \alpha_j) < \frac{\varepsilon}{2}$, for this implies that $j = 1,2 \ldots\ldots (2)$

Assume $P = \{x_0, x_1, x_2, \ldots\ldots, x_n\}$

Write $M_i = Sup\{f(x) | x \in [x_{i-1}, x_i]\}$ and

$m_i = Inf\{f(x) | x \in [x_{i-1}, x_i]\}$ for $1 \le i \le n$.

Consider $U(P, f, \alpha_1+\alpha_2) = \sum_{i=1}^{n} M_i \Delta(\alpha_1+\alpha_2)$

$= \sum_{i=1}^{n} M_i((\alpha_1+\alpha_2)(x_i) - (\alpha_1+\alpha_2)(x_{i-1}))$

$= \sum_{i=1}^{n} M_i[\alpha_1(x_i) + \alpha_2(x_i) - \alpha_1(x_{i-1}) - \alpha_2(x_{i-1})]$

$= \sum_{i=1}^{n} M_i \Delta\alpha_1 + \sum_{i=1}^{n} M_i \Delta\alpha_2$

$= U(P, f, \alpha_1) + U(P, f, \alpha_2)$

Therefore $U(P, f, \alpha_1+\alpha_2) = U(P, f, \alpha_1) + U(P, f, \alpha_2)$

Similarly $L(P, f, \alpha_1+\alpha_2) = L(P, f, \alpha_1) + L(P, f, \alpha_2)$

Now consider $U(P, f, \alpha_1+\alpha_2) - L(P, f, \alpha_1+\alpha_2)$

$= U(P, f, \alpha_1) + U(P, f, \alpha_2) - L(P, f, \alpha_1) - L(P, f, \alpha_2) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ (by 2)

So for $\varepsilon > 0$ there exists a partition P of $[a,b]$ such that

$U(P, f, \alpha_1+\alpha_2) - L(P, f, \alpha_1+\alpha_2) < \varepsilon$ and hence $f \in R(\alpha_1+\alpha_2)$

Next we will show that $\int_a^b f d(\alpha_1+\alpha_2) - \int_a^b f d\,\alpha_1 + \int_a^b f d\,\alpha_2$

Since $f \in R(\alpha_1+\alpha_2)$, we have $\int_a^b f d(\alpha_1+\alpha_2) = \underline{\int_a^b} f d(\alpha_1+\alpha_2) = \overline{\int_a^b} f d(\alpha_1+\alpha_2)$.

Consider $\int_a^b f d(\alpha_1+\alpha_2) = inf\{U(P, f, \alpha_1+\alpha_2) | P$ is a partition of $[a,b]\}$

$= inf\{U(P, f, \alpha_1) + U(P, f, \alpha_2) | P$ is a partition of $[a,b]\}$

$\ge inf\{U(P, f, \alpha_1) | P$ is a partition of $[a,b]\}$

$+ inf\{U(P, f, \alpha_2) | P$ is a partition of $[a,b]\}$

$$= \int_a^b f d\,\alpha_1 + \int_a^b f d\,\alpha_2$$

Therefore $\int_a^b f d(\alpha_1+\alpha_2) \ge \int_a^b f d\,\alpha_1 + \int_a^b f d\,\alpha_2 \ldots\ldots\ldots (3)$

Consider $\int_a^b f d(\alpha_1+\alpha_2) = Sup\{L(P, f, \alpha_1+\alpha_2) | P$ is a partition of $[a,b]\}$

$= Sup\{L(P, f, \alpha_1) + L(P, f, \alpha_2) | P$ is a partition of $[a,b]\}$

$\leq Sup\{L(P,f,\alpha_1)|P$ is a partition of $[a,b]\}$

$+ Sup\{L(P,f,\alpha_2)|P$ is a partition of $[a,b]\}$

$= \int_a^b f d\,\alpha_1 + \int_a^b f d\,\alpha_2$

Therefore $\int_a^b fd(\alpha_1 + \alpha_2) \leq \int_a^b fd\,\alpha_1 + \int_a^b fd\,\alpha_2 \ldots \ldots\ldots(4)$

From (3) and (4) $\int_a^b fd(\alpha_1 + \alpha_2) = \int_a^b fd\,\alpha_1 + \int_a^b fd\,\alpha_2$.

12.1.7 Theorem: If $f \in R(\alpha)$ on $[a,b]$ and c is a positive constant then $f \in R(c\alpha)$ and $\int_a^b fdc\alpha = c\int_a^b fd\alpha$.

Proof: Suppose $f \in R(\alpha)$ on $[a,b]$ and c is a positive constant. Since $c > 0$ and α is monotonically increasing, $c\alpha$ is monotonically increasing.

Let $\varepsilon > 0$

Since $f \in R(\alpha)$, there exists a partition P of $[a,b]$ such that

$U(P,f,\alpha) - L(P,f,\alpha) < \frac{\varepsilon}{2} \ldots\ldots\ldots(1)$

Write $M_i = Sup\{f(x)|x \in [x_{i-1,}x_i]\}$ and

$m_i = Inf\{f(x)|x \subset [x_{i-1,}x_i]\}$ for $1 \leq i \leq n$.

Consider $U(P,f,c\alpha) = \sum_{i=1}^{n} M_i \Delta c\alpha_i$

$$= \sum_{i=1}^{n} M_i(c\alpha(x_i)c\alpha(x_{i-1}))$$

$$= c\sum_{i=1}^{n} M_i(\alpha(x_i)\alpha(x_{i-1})) = c\sum_{i=1}^{n} M_i\Delta\alpha_i = cU(P,f,\alpha)$$

Therefore $U(P,f,c\alpha) = cU(P,f,\alpha)$.

Similarly $L(P,f,c\alpha) = cL(P,f,\alpha)$.

Now consider $U(P,f,c\alpha) - cL(P,f,\alpha) < c\frac{\varepsilon}{c} = \varepsilon$ (By (1))

So for $\varepsilon > 0$ there exists a partition P of $[a,b]$ such that $U(P,f,c\alpha) - L(P,f,c\alpha) < \varepsilon$ and hence $f \in R(c\alpha)$

Next we will show that $\int_a^b fdc\alpha = c\int_a^b fd\alpha$

Since $f \in R(c\alpha)$, we have $\int_a^b fdc\alpha = \underline{\int_a^b} fdc\alpha = \overline{\int_a^b} fdc\alpha$.

Consider $\int_a^b fdc\alpha = \underline{\int_a^b} fdc\alpha = Sup\{L(P,f,c\alpha)|P$ is a partition of $[a,b]\}$

$= Sup\{cL(P,f,\alpha)|P$ is a partition of $[a,b]\}$

$= Sup\{L(P,f,\alpha)|P$ is a partition of $[a,b]\}$

$= c\int_a^b fd\alpha.$

12.1.8 Theorem: If $f \in R(\alpha)$ on $[a,b]$ and $g \in R(\alpha)$ on $[a,b]$, then

(a) $fg \in R(\alpha)$

(b) $|f| \in R(\alpha)$ and $\left|\int_a^b f\,d\alpha\right| \leq \int_a^b |f|\,d\alpha$

Proof: Suppose $f \in R(\alpha)$ on $[a,b]$ and $g \in R(\alpha)$ on $[a,b]$

(a) First we show that $f^2 \in R(\alpha)$, Since f is bounded, we have $m \le f(x) \le M$ for all $x \in [a,b]$ for some real numbers m and M.

Define $\varphi: [m,M] \to \mathbb{R}$ as $\varphi(t) = t^2$ for all $[m,M]$

Then φ is continuous on $[m,M]$

Write $h = \varphi$. Then $h(x) = \varphi(f(x)) = (f(x))^2 = f^2(x)$ for all $x \in [a,b]$.

This implies $h = f^2$.

By known Theorem, $h \in R(\alpha)$ and hence $f^2 \in R(\alpha)$

Since $f, g \in R(\alpha)$ on (a,b), by Theorem 12.1.1, $f+g \in R(\alpha)$

By Theorem 12.1.1 $f-g \in R(\alpha)$

Therefore $(f+g)^2 + (f-g)^2 \in R(\alpha)$ and hence $fg \in R(\alpha)$.

(b) $\varphi: [m,M] \to \mathbb{R}$ as $\varphi(t) = |t|$ for all $t \in [m,M]$

Then φ is continuous on $[m,M]$

Write $h = [\varphi \circ f]$. Then $h(x) = \varphi(f(x)) = |f(x)| = |f|x$ for all $x \in [a,b]$.

This implies $h = f^2$

By theorem 11.1.16, $h \in R(\alpha)$ and hence $|f| \in R(\alpha)$. Choose $c = \pm 1$

So that $c\int_a^b f d(\alpha) \ge 0$

Therefore $\left|\int_a^b f\, d\alpha\right| = c\int_a^b f d(\alpha) = \int_a^b cf d\alpha \le \int_a^b |f| d\alpha$ (Since $cf \le |f|$)

So, $\left|\int_a^b f\, d\alpha\right| \le \int_a^b |f|\, d\alpha$

12.1.9 Definition: The unit step function I is defined by $I(x) = \begin{cases} 0 \text{ if } x \le 0 \\ 1 \text{ if } x > 0 \end{cases}$

12.1.10 Note: I is continuous, at every point $x \ne 0$. I is not continuous at $x = 0$.

12.1.11 Theorem: If $a < s < b$, f is bounded on $[a,b]$, f is continuous at s and $\alpha(x) = I(x-s)$, then $\int_a^b f\, d\alpha = f(s)$.

Proof: Suppose $a < s < b$, f is bounded on $[a,b]$, f is continuous at s and $\alpha(x) = I(x-s)$, for all $x \in [a,b]$.

If $x \le s$, then $\alpha(x) = I(x-s) = 0$

If $x > s$, then $\alpha(x) = I(x-s) = I$. Clearly α is not continuous at $x = s$.

First we show that $f \in R(\alpha)$ on $[a,b]$.

Let $\varepsilon > 0$. Put $\varepsilon_1 = \frac{\varepsilon}{4}$.

Since f is continuous at s, there exists $\delta > 0$ such that $|f(t) - f(s)| < \varepsilon_1$ when $t \in (a,b)$ with $|s-t| < \delta$. That is $|f(t) - f(s)| < \varepsilon_1$ whenever

$a < s - \delta < s + \delta < b \ldots\ldots\ldots..(1)$.

Write $x_0 = a, x_1 = s, x_2 = s + \frac{\delta}{2}, x = b$. Then $P = \{x_0, x_1, x_2, x_3\}$ is a partition of $[a, b]$.

Consider $\alpha(x_0) = \alpha(a) = I(a - s) = 0$

$\alpha(x_1) = \alpha(a) = I(s - s) = 0$

$\alpha(x_2) = \alpha\left(s + \frac{\delta}{2}\right) = I\left(s + \frac{\delta}{2} - s\right) = I\left(\frac{\delta}{2}\right) = I$

$\alpha(x_3) = \alpha(b) = I(b - s) = I$

Consider $\Delta(\alpha_1) = \alpha(x_1) - \alpha(x_0) = 0$

$\Delta(\alpha_2) = \alpha(x_2) - \alpha(x_1) = 0$

$\Delta(\alpha_3) = \alpha(x_3) - \alpha(x_2) = 1 - 1 = 0$

Write $M_i = Sup\{f(x)|x \in [x_{i-1,} x_i]\}$ and

$m_i = Inf\{f(x)|x \in [x_{i-1,} x_i]\}$ for $i = 1,2,3$.

$U(P, f, \alpha) = M_2$ and $L(P, f, \alpha) = m_2$

Consider $-L(P, f, \alpha) = -m_2 = -inf\{f_1(y)|y \in [x_{1,} x_2]\}$

$= sup\{f(y)|y \in [x_{1,} x_2]\}$

Now consider

$U(P, f, \alpha) - L(P, f, \alpha) = sup\{f(x)|x \in [x_{1,} x_2]\} + sup\{-f(y)|y \in [x_{1,} x_2]\}$

$= sup\{f(x) - f(y)|x, y \in [x_{1,} x_2]\} \leq \varepsilon_1 + \varepsilon$ (by (1))

$= 2\varepsilon_1 = \frac{\varepsilon}{2} < \varepsilon \ldots\ldots\ldots (2)$

Thus for $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$ and hence $f \in R(\alpha)$.

Next we will show that $\int_a^b f\, d\alpha = f(s)$.

Let P be the partition as above. Then $U(P, f, \alpha) = M_2$ and $L(P, f, \alpha) = m_2$

Consider $L(P, f, \alpha) = m_2 = inf\{f(x)|x \in [x_{1,} x_2]\} \leq f(s)$

$\leq inf\{f(x)|x \in [x_{1,} x_2]\} = U(P, f, \alpha)$

This implies $L(P, f, \alpha) \leq f(s) \leq U(P, f, \alpha) \ldots\ldots\ldots\ldots (3)$

Also we have $L(P, f, \alpha) \leq \int_a^b f\, d\alpha \leq U(P, f, \alpha) \ldots\ldots\ldots (4)$

From (3) and (4), we have $\left|f(s) - \int_a^b f\, d\alpha\right| \leq U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$ (by (2)).

This implies $\left|f(s) - \int_a^b f\, d\alpha\right| < \varepsilon$.

Since $\varepsilon > 0$ is arbitrary, we have $\int_a^b f\, d\alpha = f(s)$.

12.1.12 Theorem: Suppose $c_n \geq 0$ for $n = 1,2, \ldots\ldots$. $\sum_{n=1}^{\infty} c_n$ converges, $\{s_n\}$ is a sequence of distinct points in (a, b) and $\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n)$ and f is continuous on $[a, b]$. Then $\int_a^b f\, d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$.

Proof: First we will show that $\sum_{n=1}^{\infty} c_n I(x - s_n)$ converges.

For any $x \in [a, b]$ $|c_n I(x - s_n)| = |c_n||I(x - s_n)| \leq |c_n| = c_n$ for $n = 1,2,3 \ldots\ldots$

Since $\sum_{n=1}^{\infty} c_n$ converges, by the comparison test $\sum_{n=1}^{\infty} c_n I(x - s_n)$ converges.

Next we will show that α is monotonically increasing. Suppose $x, y \in [a, b]$ such that $x - s_n \leq y - s_n$ for all n. This implies $I(x - s_n) \leq I(y - s_n)$ and hence
$\sum_{n=1}^{\infty} c_n I(x - s_n) \leq \sum_{n=1}^{\infty} c_n I(y - s_n)$
That is $\alpha(x) \leq \alpha(y)$.
So α is monotonically increasing.
Since f is continuous on $[a, b]$ by Theorem 11.1.12, $f \in R(\alpha)$
Since $a < s_n$ for all n, $\alpha(a) = \sum_{n=1}^{\infty} c_n I(a - s_n) = 0$
Since $s_n < b$ for all n, $\alpha(b) = \sum_{n=1}^{\infty} c_n I(b - s_n) = 0$
Since f is continuous on $[a, b]$, f is bounded on $[a, b]$.
So put $M = sup\{f(x) | x \in [a, b]\}$.
Now we will show that $\sum_{n=1}^{\infty} c_n I(b - s_n) = \sum_{n=1}^{\infty} c_n$. That is, we have to show that the sequence of partial sums of the series $\sum_{n=1}^{\infty} c_n f(s_n)$ converges to $\int_a^b f\, d\alpha$.
Let $\varepsilon > 0$. write $\varepsilon_1 = \frac{\varepsilon}{M+1}$
Since $\sum_{n-1}^{\infty} c_n$ converges, there exists a positive integer
N such that $\sum_{n=N+1}^{\infty} c_n < \varepsilon_1 \ldots\ldots\ldots (1)$
Put $\alpha_1(x) = \sum_{n=1}^{\infty} c_i I(x - s_i)$ and $\sum_{n=N+1}^{\infty} c_i I(x - s_i)$ for all $x \in [a, b]$.
Then $\alpha = \alpha_1 + \alpha_2$ and α_1 and α_2 are monotonically increasing on $[a, b]$ since f is continuous on $[a, b]$ by Theorem 11.1.12, $f \in R(\alpha_1)$ and $f \in R(\alpha_2)$
For $i = 1,2$. Put $I_i(x) = I(x - s_i)$ for all $x \in [a, b]$.
Since $a < s_i < b$ and f is continuous at s_i and f is bounded on $[a, b]$ by Theorem 11.1.11.
$\int_a^b f\, d\alpha_i = f(s_i)$ for $i = 1,2, \ldots\ldots\ldots (2)$
By Theorem 11.1.6 and Theorem 11.1.7
$\int_a^b f\, d\alpha_1 = \int_a^b f\, d(\sum_{n=1}^{N} c_i) = \sum_{n=1}^{N} \int_a^b f\, d(c_i\alpha_i) = \sum_{n=1}^{N} c_i \int_a^b f\, d\alpha_i = \sum_{i=1}^{N} c_i f(s_i)$
By (2)
Consider $\alpha_2(b) - \alpha_2(a) = \sum_{i=N+1}^{\infty} c_i I(b - s_i) = \sum_{i=N+1}^{\infty} c_i < \varepsilon_i$ (By(1))
By Theorem 11.1.5, $\left|\int_a^b f\, d\alpha_2\right| \leq M(\alpha_2(b) - \alpha_2(a)) < M\varepsilon_1$
Therefore $\left|\int_a^b f\, d\alpha - \sum_{i=1}^{N} c_i f(s_i)\right| = \left|\int_a^b f\, d\alpha - \int_a^b f\, d\alpha_1\right|$

$$= \left|\int_a^b f\, d\alpha_2\right| < M\varepsilon_1 < (M+1)\varepsilon_1 = \varepsilon$$

This implies that $\left|\int_a^b f\, d\alpha - \sum_{i=1}^{N} c_i f(s_i)\right| < \varepsilon$ for all $n \geq N$

This shows that the sequence of partial sums of the series $\sum_{i=1}^{N} c_n f(s_n)$ converges to $\int_a^b f\, d\alpha$.

Hence $\int_a^b f\, d\alpha = \sum_{i=1}^{N} c_n f(s_n)$.

12.1.13 **Note:** Let $f: [a, b] \to \mathbb{R}$ be defined $f(x) = k$ for some constant k and for all $x \in [a, b]$.
Then $f \in R$ on $[a, b]$ and $\int_a^b f(x) dx = k(b - a)$.
For this let $P = \{x_0, x_1, \ldots\ldots, x_n\}$ be a partition of $[a, b]$.

Write $M_i = Sup\{f(x) | x \in [x_{i-1}, x_i]\}$ and
$m_i = Inf\{f(x) | x \in [x_{i-1}, x_i]\}$ for $1 \le i \le n$
Then $M_i = k$ and $m_i = k$ for $1 \le i \le n$
$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} k \Delta x_i = k(b-a)$
Similarly, $U(P, f) = k(b-a)$
Therefore $U(P, f) - L(P, f) = 0 < \varepsilon$ for any $\varepsilon > 0$ and hence $f \in R$ on $[a, b]$
Consider $k(b-a) = L(P, f) \le \int_a^b f(x)dx \le U(P, f) = k(b-a)$
Therefore $\int_a^b f(x)dx = k(b-a)$.

12.1.14 Theorem : Assume α increases monotonically on $[a, b]$ and $\alpha' \in R$ on $[a, b]$. Let f be a bounded real function defined on $[a, b]$ Then $f \in R(\alpha)$ on $[a, b]$ if and only if $\alpha' \in R$. In that case $\int_a^b f(x)dx = k(b-a)$.

Proof: Suppose α increases monotonically on $[a, b]$ and also assume that f be a bounded real function defined on $[a, b]$.
Let $\varepsilon > 0$
Since $\alpha' \in R$ there exists a partition $P = \{x_0, x_1, \dots\dots, x_n\}$ of $[a, b]$ such that
$U(P, \alpha') - L(P, \alpha') < \varepsilon \dots\dots\dots (1)$
Since α' exists, α is differentiable on $[a, b]$. Then α is continuous on $[a, b]$ and α is differentiable on (a, b). This implies α is continuous on $[x_{i-1}, x_i]$ and α is differentiable on (x_{i-1}, x_i) for $1 \le i \le n$. So by mean value theorem, there exists a point $t_i \in (x_{i-1}, x_i)$ such that $\alpha(x_i) - \alpha(x_{i-1}) = \alpha'(t_i)(x_{i-1}, x_i)$ for $1 \le i \le n$.
That is $\Delta\alpha_i = \alpha'(t_i)\Delta x_i$ for $1 \le i \le n$
Since f is bounded on $[a, b]$. Put $M = Sup\{f(x) \mid x \in [a, b]\}$
Now we will show that $U(P, f, \alpha) \le U(P, f, \alpha') + M\varepsilon$
$U(P, f, \alpha') \ge U(P, f, \alpha) + M\varepsilon$
$L(P, f, \alpha) \le L(P, f, \alpha') + M\varepsilon$
$L(P, f, \alpha') \ge L(P, f, \alpha) + M\varepsilon$
Let $s_i \in (x_{i-1}, x_i)$ for $1 \le i \le n$. Then by Theorem 11.1.10 and by (1),

$$\sum_{i=1}^{n} |\alpha'(s_i) - \alpha'(t_i)| \Delta x_i < \varepsilon \dots\dots\dots (2)$$

Consider $|\sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\Delta x_i|$

$$= \left| \sum_{i=1}^{n} f(s_i)\alpha'(t_i)\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i \right|$$

$$= \left| \sum_{i=1}^{n} f(s_i)[\alpha'(t_i) - \alpha'(s_i)]\, \Delta x_i \right|$$

$$\leq \sum_{i=1}^{n} |f(s_i)||\alpha'(t_i) - \alpha'(s_i)|\Delta x_i$$

$$\leq \sum_{i=1}^{n} M\,|\alpha'(t_i) - \alpha'(s_i)|\Lambda x_i$$

$$= M\sum_{i=1}^{n} |\alpha'(t_i) - \alpha'(s_i)|\Delta x_i < M\varepsilon$$

This implies that $\sum_{i=1}^{n} f(s_i)\Delta\alpha_i \leq \sum_{i=1}^{n}(f\alpha')(s_i)\Delta x_i + M\varepsilon \ldots\ldots\ldots(3)$ and $\sum_{i=1}^{n} f(f\alpha')(s_i)\Delta x_i \leq \sum_{i=1}^{n} f(s_i)\Delta\alpha_i + M\varepsilon \ldots\ldots\ldots.(4)$

Write $M_1^* = Sup\{(f\alpha')(x)|x \in [x_{i-1,}x_i]\}$ and for $1 \leq i \leq n$

Then from (3), $\sum_{i=1}^{n} f(s_i)\Delta\alpha_i \leq \sum_{i=1}^{n}(f\alpha')(s_i)\Delta x_i + M\varepsilon$

$$\leq \sum_{i=1}^{n}{}' M_1^*\Delta x_i + M\varepsilon = U(P,f,\alpha') + M\varepsilon$$

This implies that $\sum_{i-1}^{n} f(s_i)\Delta\alpha_i < U(P,f,\alpha') + M\varepsilon \ldots\ldots\ldots.(5)$

Write $M_i = Sup\{f(x)|x \in [x_{i-1,}x_i]\}$ and

$m_i = Inf\{f(x)|x \in [x_{i-1,}x_i]\}$ for $1 \leq i \leq n$

Then from (3), $L(P,f,\alpha) = \sum_{i=1}^{n} M_i\Delta\alpha_i \leq \sum_{i=1}^{n} f(s_i)\Delta\alpha_i \leq \sum_{i=1}^{n}(f\alpha')(s_i)\Delta x_i + M\varepsilon$

This implies that $L(P,f,\alpha') - M\varepsilon \leq \sum_{i=1}^{n}(f\alpha')(s_i)\Delta x_i \ldots\ldots\ldots(6)$

Therefore inequalities (5) and (6) are true for any $s_i \in [x_{i-1,}x_i]$ for $1 \leq i \leq n$.

Consider $U(P,f,\alpha) = \sum_{i=1}^{n} M_i\Delta\alpha_i = M_1\Delta\alpha_1 + M_2\Delta\alpha_2 + \cdots\ldots.+M_n\Delta\alpha_n$

$= Sup\{f(x)|x \in [x_{i-1,}x_i]\}\Delta\alpha_1 + \cdots + Sup\{f(x)|x \in [x_{i-1,}x_i]\}\Delta\alpha_n$

$$= \sum_{i=1}^{n} Sup\{f(x)\Delta\alpha_i|x \in [x_{i-1,}x_i]\}$$

Therefore $U(P,f,\alpha) = Sup\sum_{i=1}^{n}\{f(x)\Delta\alpha_i|s_i \in [x_{i-1,}x_i]\} \ldots\ldots\ldots\ldots.(7)$

Similarly $U(P,f,\alpha) = Inf\sum_{i=1}^{n}\{f(\alpha')(s_i)\Delta\alpha_i|s_i \in [x_{i-1,}x_i]\} \ldots\ldots\ldots\ldots.(8)$

From (5) $U(P,f,\alpha') + M\varepsilon$ is an upper bound of $\sum_{i=1}^{n}\{f(s_i)\Delta\alpha_i|s_i \in [x_{i-1,}x_i]\}$ and From (6) $L(P,f,\alpha) - M\varepsilon$ is a lower bound of $\sum_{i=1}^{n}\{f(\alpha')(s_i)\Delta\alpha_i|s_i \subset [x_{i-1,}x_i]\}$.

From (7) and (8) $U(P,f,\alpha) \leq U(P,f,\alpha') + M\varepsilon$ and

$L(P,f,\alpha) - M\varepsilon \leq U(P,f,\alpha')$

Therefore $U(P,f,\alpha) \leq U(P,f,\alpha') + M\varepsilon \ldots\ldots\ldots(9)$

$L(P,f,\alpha) \leq U(P,f,\alpha') - M\varepsilon \ldots\ldots\ldots\ldots.(10)$

Similarly from (4) we can show that $U(P,f,\alpha') \leq U(P,f,\alpha) + M\varepsilon \ldots\ldots\ldots.(11)$

$L(P,f,\alpha') \leq L(P,f,\alpha) + M\varepsilon \ldots\ldots\ldots..(12)$

Now, we will show that $f \in R(\alpha)$ on $[a,b]$ if and only if $\alpha' \in R$ on $[a,b]$

Suppose $f \in R(\alpha)$ on $[a,b]$.

Let $\varepsilon > 0$. Put $\varepsilon_1 = \frac{\varepsilon}{2M+1}$

Since $\alpha' \in R$ on $[a,b]$ there exists a partition P_1 of $[a,b]$ such that $U(P_1,\alpha') - L(P_1,\alpha') < \varepsilon_1$

Since $f \in R(\alpha)$ on $[a, b]$, there exists a partition P_2 of $[a, b]$ such that

$U(P_2, f, \alpha') \quad L(P_2, f, \alpha') < \varepsilon_1$

Write $P = P_1 \cup P_2$. Then P is a partition of $[a, b]$ and P is the common refinement of P_1 and P_2.

Then by Theorem 11.1.6.

$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon_1$ and $U(P_1, \alpha') - L(P_1, \alpha') < \varepsilon_1$

This implies that P satisfies (9), (10), (11) and (12)

Consider $U(P, f, \alpha') \leq U(P, f, \alpha') + M\varepsilon_1$

$L(P, f, \alpha) \leq U(P, f, \alpha') + M\varepsilon_1$

From the above two in equalities

$U(P, f, \alpha') - U(P, f, \alpha') \leq U(P, f, \alpha') - L(P, f, \alpha') + M\varepsilon_1 + M\varepsilon_1 < 2M\varepsilon_1 + \varepsilon_1 = \varepsilon.$

Therefore for $\varepsilon > 0$

there exists a partition P of $[a, b]$ such that $U(P, f, \alpha') - L(P, f, \alpha') < \varepsilon$

and hence $f\alpha' \in R$ on $[a, b]$.

Conversely $f\alpha' \in R$ on $[a, b]$.

Let $\varepsilon > 0$

Put $\varepsilon_1 = \frac{\varepsilon}{2M+1}$

Since $\alpha' \in R$ on $[a, b]$ there exists a partition P_1 of $[a, b]$ such that $U(P_1, \alpha') - L(P_1, \alpha') < \varepsilon_1$

Since $f\alpha' \in R$ on $[a, b]$, there exists a partition P_2 of $[a, b]$ such that

$U(P_2, f, \alpha') - L(P_2, f, \alpha') < \varepsilon_1$

Write $P = P_1 \cup P_2$. Then P is a partition of $[a, b]$ and P is the common refinement of P_1 and P_2.

Then by Theorem 11.1.6.

$U(P_1, \alpha') - L(P_1, \alpha') < \varepsilon_1$ and $U(P, f, \alpha') - L(P, f, \alpha') < \varepsilon_1$

This implies that P satisfies (9), (10), (11) and (12)

Now consider,

$U(P, f, \alpha) - L(P, f, \alpha) \leq U(P, f, \alpha') - L(P, f, \alpha') + M\varepsilon_1 + M\varepsilon_1 < 2M\varepsilon_1 + \varepsilon_1 = \varepsilon.$

Thus for $\varepsilon > 0$

there exists a partition P of $[a, b]$ such that $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$

and hence $f \in R(\alpha)$ on $[a, b]$.

Now we will show that $\int_a^b f\, d\alpha = \int_a^b (f\alpha')(x)dx = \int_a^b f(x)\alpha'(x)dx$

Let $\varepsilon > 0$

Put $\varepsilon_1 = \frac{\varepsilon}{M+1}$

Since $\alpha' \in R$ on $[a, b]$ there exists a partition Q of $[a, b]$ such that $U(Q, \alpha') - L(Q, \alpha') < \varepsilon_1$

Let S be any partition of $[a, b]$. Put $P - S \cup Q$. Then P is the common refinement of S and Q and

$U(P_1, \alpha') - L(P_1, \alpha') \leq U(Q, \alpha') - L(Q, \alpha') < \varepsilon_1$

Now, P satisfies (9), (10), (11) and (12) for ε_1

Consider $\int_a^b f\, d\alpha \leq U(P, f, \alpha) \leq U(P, f, \alpha') + M\varepsilon_1$

$\leq U(S, f, \alpha') + M\varepsilon_1 < U(S, f, \alpha') + M\varepsilon_1 + \varepsilon_1$

$= U(S, f, \alpha') + \varepsilon$

This implies that $\int_a^b f\, d\alpha < U(S, f, \alpha') + \varepsilon$ for any partition S of $[a, b]$.

Consider $\int_a^b f\,d\alpha \geq L(P,f,\alpha) \leq L(P,f,\alpha') - M\varepsilon_1$
$\geq L(S,f,\alpha') - M\varepsilon_1 > L(S,f,\alpha') - M\varepsilon_1 - \varepsilon_1$
$= L(S,f,\alpha') - \varepsilon$

Therefore $\int_a^b f\,d\alpha > L(S,f,\alpha') - \varepsilon$ for any partition S of $[a,b]$.

Now $\int_a^b f\,d\alpha - \varepsilon \leq inf\{U(S,f,\alpha')|S \text{ is a partition of } [a,b]\} = \int_a^b (f\alpha')(x)dx$

And $\int_a^b f\,d\alpha + \varepsilon \geq Sup\{L(S,f,\alpha')|S \text{ is a partition of } [a,b]\} = \int_a^b (f\alpha')(x)dx$

Therefore $\int_a^b f\,d\alpha + \varepsilon \leq \int_a^b (f\alpha')(x)dx \leq \int_a^b f\,d\alpha + \varepsilon$

This implies $\left|\int_a^b f\,d\alpha - \int_a^b (f\alpha')(x)dx\right| \leq \varepsilon$

Since $\varepsilon > 0$ is arbitrary, we have $\int_a^b f\,d\alpha = \int_a^b (f\alpha')(x)dx$.

12.1.15 Theorem (Change of variable): Suppose φ is a strictly increasing continuous function that maps an interval $[A,B]$ onto $[a,b]$. Suppose α is monotonically increasing on $[a,b]$ and $f \in R(\alpha)$ on $[a,b]$. Define β and g on $[A,B]$ by $\beta(y) = \alpha(\varphi(y))$, $g(y) = f(\varphi(y))$. Then $g \in R(\alpha)$ and $\int_A^B g\,d\beta = \int_a^b f\,d\alpha$.

Proof: Since $f \in R(\alpha)$, f is a bounded function and so $f[a,b]$ is bounded

Since φ is onto, $g[A,B] = f(\varphi[A,B]) = f[a,b]$

This implies that $g[A,B]$ is bounded, hence g is bounded.

Let $y_1, y_2 \in [A,B]$ be such that $y_1 \leq y_2$.

Since φ is increasing on $[A,B]$, $\varphi(y_1) \leq \varphi(y_2)$

Since α is monotonically increasing on $[a,b]$, we have $\alpha(\varphi(y_1)) \leq \alpha(\varphi(y_2))$.

This implies that $\beta(y_1) \leq \beta(y_2)$ and hence β is monotonically increasing on $[A,B]$.

Next we will prove that $\varphi(A) = a$ and $\varphi(B) = b$.

Clearly $\varphi(A) \in [a,b]$, This implies that $a \leq \varphi(A)$.

Since φ is onto and $a \in [a,b]$, there exists $y \in [A,B]$ such that $\varphi(y) = a$.

If $A < y$, then $\varphi(A) < \varphi(y)$ (Since φ is strictly increasing).

This implies that $\varphi(A) < a$, a contradiction.

So $A = y$ and hence $\varphi(A) = a$.

Similarly, we can show that $\varphi(B) = b$.

Let $Q = \{y_0, y_1, \dots \dots, y_n\}$ be a partition of $[A,B]$.

Then $y_0 = A, y_1 = B$ and $y_0 \leq y_1 \leq \cdots \leq y_n$.

This implies that $\varphi(A) \leq \varphi(y_1) \leq \cdots \leq \varphi(y_n) = \varphi(B)$

Take $x_i - \varphi(y)\ for\ 0 \leq i \leq n$ then

$a = x_1 \leq x_2 \leq \cdots < x_n = b$. So $P = \{x_1, x_2, \dots \dots, x_n\}$ be a partition of $[a,b]$ such that $\varphi(y_1) = x_i\ for\ 0 \leq i \leq n$.

Conversely Let $P = \{x_1, x_2, \dots \dots, x_n\}$ be a partition of $[a,b]$.

Then $a = x_1 \leq x_2 \leq \cdots < x_n = b$

Since φ is onto, for each x_i, there exists $y_i \in [A,B]$ such that $\varphi(y_i) = x_i$.

This implies that
$\varphi(y_0) = a$ and $\varphi(y_n) = b$. Since φ is strictly increasing, we have φ is one-one.
Since $\varphi(A) = a$ and $\varphi(B) = b$, we have $A = y_0, B = y_n$.
Also $A = y_0 \leq y_1 \leq \cdots \leq y_n = B$.
So $Q = \{y_0, y_1, \dots \dots, y_n\}$ is a partition of $[A, B]$ such that $\varphi(y_i) = x_i \; for \; 0 \leq i \leq n$.
Next we will prove that $f[a, b] = g[A, B]$
Let $x \in f[a, b]$. Then $x = f(y)$ for some $y \in [a, b]$.
Since φ is onto, there exists $t \in [A, B]$ such that $\varphi(t) = y$.
Consider $g(t) = f(\varphi(t)) = f(y) = x$. This implies that $x \in g[A, B]$.
So, $f[a, b] \subseteq g[A, B]$.
Let $y \in g[A, B]$. Then $y \in g(t)$ for some $t \in [A, B]$.
Now $\varphi(t) \in [a, b]$. This implies that $f(\varphi(t)) \in [a, b]$.
Since $g(t) = f(\varphi(t))$, we have $g(t) \in f[a, b]$ and so $f \in [a, b]$.
Hence $f \in [a, b] - g[A, B]$
Let $P = \{x_1, x_2, \dots \dots, x_n\}$ be a partition of $[a, b]$ then there exists a partition
$Q = \{y_0, y_1, \dots \dots, y_n\}$ of $[A, B]$ such that $\varphi(y_i) = x_i \; for \; 0 \leq i \leq n$.
This implies that $f[x_{i-1}, x_i] = g[y_{i-1}, y_i]$ for $1 \leq i \leq n$.
Write $M_i - Sup\{f(x) | x \in [x_{i-1}, x_i]\}$ and
$m_i = Inf\{f(x) | x \in [x_{i-1}, x_i]\}$ and
Write $N_i = Sup\{g(y) | y \in [y_{i-1}, y_i]\}$ and
$n_i = Inf\{g(y) | y \in [y_{i-1}, y_i]\}$
For $1 < i < n$, Consider
$M_i = Sup\{f(x) | x \in [x_{i-1}, x_i]\} = sup f[x_{i-1}, x_i] = Sup \; g[y_{i-1}, y_i] = N_i$
This implies that $M_i = N_i$ for $1 \leq i \leq n$. Similarly $m_i = n_i$ for $1 \leq i \leq n$
Consider $U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i = \sum_{i=1}^{n} M_i(\alpha(x_i) - \alpha(x_{i-1}))$
$= \sum_{i=1}^{n} M_i \left(\alpha(\varphi(y_1)) - \alpha(\varphi(y_{i-1}))\right) = \sum_{i=1}^{n} N_i(\beta(y_i) - \beta(y_{i-1})) = U(Q, g, \beta)$.
Therefore $U(P, f, \alpha) = U(Q, g, \beta)$
Similarly we can show that $L(P, f, \alpha) = U(Q, g, \beta)$
Let $\varepsilon > 0$
Since $f \in R(\alpha)$, there exists a partition P of $[A, B]$ such that
$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon \dots \dots \dots (1)$
Since P is a partition of $[a, b]$ by the above facts, we have a partition Q of $[A, B]$ such that
$U(P, f, \alpha) = U(Q, g, \beta)$ and $L(P, f, \alpha) = U(Q, g, \beta)$.
Then by (1) $U(Q, g, \beta) - L(Q, g, \beta) < \varepsilon$
Therefore $g \in R(\alpha)$ on $[A, B]$.
Consider $\int_a^b f \, d\alpha - Sup\{L(P, f, \alpha) | P$ is a partition of $[a, b]\}$
$= Sup\{L(Q, g, \beta) | P$ is a partition of $[A, B]\}$
$= \int_A^B g \, d\beta$

Hence $\int_a^b f\, d\alpha = \int_A^B g\, d\beta$.

Short Answer Questions

1. If $f_1 \in R(\alpha)$ and $f_2 \in R(\alpha)$ on $[a, b]$ and $f_1(x) \leq f_2(x)$ on $[a, b]$, then show that $\int_a^b f_1\, da = \int_A^B f_2\, d\alpha$.

2. Define the unit step function I and show that I is continuous at every point $x \neq 0$.

3. Let $f: [a, b] \rightarrow R$ be defined by $f(x) = k$ for some constant k and for all $x \in [a, b]$.

Model Examination Questions

1. If $f \subset R(\alpha)$ on $[a, b]$ and $a < c < b$, then $f \subset R(\alpha)$ on $[a, c]$ and on $[c, b]$ and $\int_a^c f\, d\alpha + \int_b^c f\, d\alpha = \int_a^b f\, d\alpha$.

2. If $a < s < b$, f is bounded on $[a, b]$, f is continuous at s and $\alpha(x) = I(x - s)$, then $\int_a^b f\, d\alpha = f(s)$.

3. Suppose φ is a strictly increasing continuous function that maps an interval $[A, B]$ onto $[a, b]$. Suppose α is monotonically increasing on $[a, b]$ and $f \in R(\alpha)$ on $[a, b]$. Define β and g on $[A, B]$ by $\beta(y) = \alpha(\varphi(y))$, $g(y) = f(\varphi(y))$. Then $g \in R(\alpha)$ and $\int_A^B g\, d\beta = \int_a^b f\, d\alpha$.

Exercises

1. Suppose f is a bounded real function on $[a, b]$ and $f^2 \in R\ [a, b]$. Does it follow that $f \in R(\alpha)$ Does the answer change if we assume that $f^3 \in R$?

Answers to Short Answer Questions

For **1**, see theorem **12.1.3**.

For **2**, see definition **12.1.9**

For **3**, see note **12.1.13**

12.2 SUMMARY:

This lesson provides a comprehensive introduction to the Riemann-Stieltjes integral, covering its definition, properties, and applications. Learners will develop a deep understanding of the integral's properties and learn how to compute it for various functions. The Lesson Components are Introduction to the Riemann-Stieltjes integral, Definitions and properties of the integral, Theorems with proofs, Exercise problems to reinforce understanding.

12.3 TECHNICAL TERMS:

- Partition
- Riemann Stieltjes Integral
- Commom Refinement
- Monotonically increasing
- Bounded
- Continuous
- Unit Step function
- Converges
- Sequence and series.

12.4 SELF ASSESSMENT QUESTIONS:

1. If $a < s < b$, f is bounded on $[a, b]$, f is continuous at s and $\alpha(x) = I(x - s)$, then $\int_a^b f\, d\alpha = f(s)$.
2. Suppose φ is a strictly increasing continuous function that maps an interval $[A, B]$ onto $[a, b]$. Suppose α is monotonically increasing on $[a, b]$ and $f \in R(\alpha)$ on $[a, b]$. Define β and g on $[A, B]$ by $\beta(y) = \alpha(\varphi(y))$, $g(y) = f(\varphi(y))$. Then $g \in R(\alpha)$ and $\int_A^B g\, d\beta - \int_a^b f\, d\alpha$.
3. Suppose f is a bounded real function on $[a, b]$ and $f^2 \in R\ [a, b]$. Does it follow that $f \in R(\alpha)$ Does the answer change if we assume that $f^3 \in R$?

12.5 SUGGESTED READINGS:

1. Principles of Mathematical Analysis, Third Edition, Mc Graw-Hill International Editions Walter Rudin.

2. Mathematical Analysis by Tom M. Apostol, Narosa Publishing House, 2nd Edition, 1985.

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LESSON-13
INTEGRATION AND DIFFERENTIATION

OBJECTIVES:

The objective of the lesson is to help the learners to understand

- To prove integrations and differentiation are (in a certain sense) inverse operations and prove a selection of theorems concerning integration.
- To understand the relationship between integration and differentiations.

STRUCTURE:

13.0 INTRODUCTION:

In this lesson, it has been shown that integration and differentiation are, in certain sense, inverse operations. The fundamental theorem of calculus and integration by parts are proved. Also the integration of vector valued function is studied. (Further rectifiable curve is defined and it is proved that every continuously differentiable curve on $[a, b]$ is rectifiable).

13.1 INTEGRATIONAND DIFFERENTIATION:

13.1.1 Theorem: Let f be a real valued function on $[a, b]$ such that $f \in R[a, b]$. For $a \leq x \leq b$, put $F(x) = \int_a^x f(t)\,dt$. Then F is continuous on $[a, b]$; furthermore, if f is continuous at a point x_0 of $[a, b]$, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

Proof: Given that f *is* a real valued function defined on $[a, b]$ such that $f \in R[a, b]$.

Also given that for $\leq x \leq b$, $F(x) = \int_a^x f(t)\,dt$.

Since $f \in R[a, b]$ is bounded on $[a, b]$. Then there exists an M such that $|f(t)| \leq M$ for all $t \in [a, b)$.

Let $\varepsilon > 0$. Write $\delta = \frac{\varepsilon}{M+1}$, Then $\delta > 0$.

Let $x, y \in [a, b]$ such that $x < y$ and $|x - y| < \delta$.

Consider $|F(x) - F(y)| = \left|-\int_a^x f(t)\,dt - \int_a^y f(t)\,dt\right|$

$\left|-\int_x^y f(t)\,dt\right| = \left|\int_x^y f(t)\,dt\right| \leq M(y - x)$ (By theorem)

$= M(x - y) < M\delta < (M + 1)\delta < \varepsilon.$

So for $\varepsilon > 0$, there exists $\delta > 0$ such that $|F(x) - F(y)| < \varepsilon$, whenever $|x - y| < \delta$.

This implies that F is uniformly continuous and hence F is continuous on $[a, b]$.
Suppose f is continuous at a point $x_0 \in [a, b]$.
Now we will show that F is differentiable at x_0 and $F'(x_0) = f(x_0)$.
Define $h(t) = \frac{F(t)-F(x_0)}{t-x_0}$ for all t such that $a < t < b$ and t$\neq x_0$.
Now we show that $\lim_{t\to x_0} h(t) = f(x_0)$
Let $\varepsilon > 0$. Since f is continuous at x_0, there exists a $\delta > 0$ such that $|f(x_0) - f(t)| < \varepsilon$ whenever $t \in [a, b]$ with $|x_0 - t| < 0$ … … . (1)
Suppose $0 < |t - x_0| < 0$. Then $x_0 - \delta < t < x_0 + \delta$.
This implies that either
$x_0 - \delta < x_0 < t < x_0 + \delta$ or $x_0 - \delta < t < x_0 < x_0 + \delta$.
Suppose $x_0 - \delta < t < x_0 < x_0 + \delta$.
Consider $|h(t) - f(x_0)| = \left|\frac{F(t)-F(x_0)}{t-x_0} - f(x_0)\right|$

$$= \frac{1}{x_0-t}\left|\int_x^{x_0} f(u)\,du - \int_a^t f(u)\,du - f(x_0)(x_0-t)\right|$$
$$= \frac{1}{x_0-t}\left|\int_t^{x_0} f(u)\,du - f(x_0)(x_0-t)\right|$$
$$= \frac{1}{x_0-t}\left|\int_t^{x_0} f(u)\,du - \int_t^{x_0} f(x_0)\,du\right|$$
$$= \frac{1}{x_0-t}\left|\int_t^{x_0}(f(u) - f(x_0))du\right|$$
$$< \frac{1}{x_0-t}\varepsilon(x_0-t) = \varepsilon \text{ (by (1))}$$

Therefore $|h(t) - f(x_0)| < \varepsilon$
Similarly we can show that $x_0 - \delta < t < x_0 < x_0 + \delta$, then $|h(t) - f(x_0)| < \varepsilon$.
So, $\lim_{t\to x_0} h(t) = f(x_0)$. That is $\lim_{t\to x_0} \frac{F(t)-F(x_0)}{t-x_0} = f(x_0)$.
This shows that F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

13.1.2 Theorem: (The Fundamental theorem of Calculus):

If $f \in R[a, b]$ and if there is a differentiable function F on $[a, b]$ such that $F' = f$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Proof: Suppose $f \in R[a, b]$ and F is a differentiable function on $[a, b]$ such that $F' = f$. Let ε be any positive real number.
Since $f \in R[a, b]$, there exists a partition $P = \{x_0, x_1, x_2, \dots, x_n\}$ of $[a, b]$ such that $U(P, f) - L(P, f) < \varepsilon$ … … … . . (1),
Since F is differentiable on $[a, b]$, F is differentiable on $[x_{i-1,} x_i]$ for $1 \le i \le n$.
This implies that F is differentiable on $(x_{i-1,} x_i)$ and F is continuous on $[x_{i-1,} x_i]$ for $1 \le i \le n$.
By Mean Value theorem, there exists $t_i \in (x_{i-1,} x_i)$ such that
$F(x_i) - F(x_{i-1}) = f(t_i)(x_i - x_{i-1})$ for $1 \le i \le n$.
Since $F' = f$ on $[a, b]$. We have $F(x_i) - F(x_{i-1}) = f(t_i)\Delta x_i$ for $1 \le i \le n$.
Now $\sum_{i=}^n f(t_i)\Delta x_i = \sum_{i=}^n \left(F(x_i) - F(x_{i-1})\right) = F(b) - F(a)$.
Therefore $L(P, f) = \sum_{i=1}^n m_i \Delta x_i \le \sum_{i=1}^n f(t_i)\Delta x_i \le \sum_{i=1}^n M_i \Delta x_i = U(P, f)$

Where $M_i = Sup\{f(x)|x \in [x_{i-1,}x_i]\}$ and $m_i = Inf\{f(x)|x \in [x_{i-1,}x_i]\}$ for $1 \le i \le n$.

So $L(P, f) \le F(b) - F(a) \le U(P, f)$…………………..(2)

Also $L(P, f) \le \int_a^b f(x)\,dx \le U(P, f)$…………………..(3)

From (1), (2) and (3), $\left|F(b) - F(a) - \int_a^b f(x)dx\right| < \varepsilon$.

Since $\varepsilon > 0$ is arbitrary $\int_a^b f(x)\,dx = F(b) - F(a)$.

13.1.3 Theorem: (Integration by parts): Suppose F and G are differentiable functions on $[a, b]$ $F' = f \in R$ and $G' = g \in R$ then

$\int_a^b F(x)g(x)\,dx = F(b)g(b)F(a)G(a) - \int_a^b f(x)G(x)dx$.

Proof: Suppose F and G are differentiable functions on $[a, b]$ $F' = f \in R$ and $G' = g \in R$.

Define H on $[a, b]$ as $H(x) = F(x)G(x)$ for any $x \in [a, b]$.

Since F and G are differentiable on $[a, b]$, H is also differentiable on $[a, b]$ and

$$H' = F'G + G'F' = fG + gF.$$

Since G is differentiable on $[a, b]$, G is continuous on $[a, b]$.

Then by Known theorem, $G \in R$. Therefore $fG \in R$ similarly $gF \in R$.

By theorem 12.1.1. $fG + gF \in R$; That is $H' \in R$.

Put $h = H'$. By theorem 13.1.2 $\int_a^b h(x)\,dx = H(b) - H(a)$.

But $\int_a^b h(x)\,dx = \int_a^b (f(x)G(x) + g(x)F(x))\,dx$

$= \int_a^b f(x)G(x) + \int_a^b g(x)F(x)\,dx$

Therefore, $\int_a^b h(x)\,dx = \int_a^b f(x)G(x) + \int_a^b g(x)F(x)\,dx$

$= F(b)G(b) - F(a)G(a)$

And hence $\int_a^b F(x)g(x)\,dx = F(b)g(b)F(a)G(a) - \int_a^b f(x)G(x)dx$.

13.2 SOME MORE EXAMPLES WITH SOLUTIONS:

13.2.1 Example : Suppose f is a bounded real function on $[a, b]$ and $f^2 \in R$ on $[a, b]$. Does it follow that $f \in R$? Does the answer change if we assume that $f^3 \in R$?

Solution. The integrability of f^2 does not imply the integrability of f.

For example, one could let $f(x) = -1$ if x is irrational and $f(x) = 1$ if x is rational.

Then every upper Riemann sum of f is $b - a$ and every lower sum is $a - b$.

However, f^2, being the constant function 1, is integrable.

The integrability of f^3 does imply the integrability of f, By Known theorem with $\varphi(u) = \sqrt[3]{u}$.

13.2.2 Example : Suppose f is a real function on $[0,1]$ and $f \in R$ on $[c, 1]$ for every $c > 0$. Define $\int_0^1 f(x)\,dx = \lim_{c\to 0+} \int_c^1 f(x)\,dx$ if this limit exists (and is finite)

(a) If $f \in R$ on $[0,1]$ show that this definition of the integral agree.

(b) Construct a function f such that the above limit exists, although it fails to exist with $|f|$ in place of f.

Solution. Suppose $f \in R$ on $[0,1]$.

Let $\epsilon > 0$ be given

And let $M = \sup\{|f(x): 0 \le x \le 1|\}$.

Let $c \in \left(0, \frac{\epsilon}{4M}\right]$ be fixed,

And consider any partition of $[0,1]$ containing c for which the upper and lower Riemann sums $\sum M_j \left(t_j - t_{j-1}\right)$ and $\sum m_j \left(t_j - t_{j-1}\right)$ of f differ by less than $\frac{\epsilon}{4}$.

Then the partition of $[c, 1]$ formed by the points of this partition that lie in this interval certainly has the property that its upper and lower Riemann sums$\sum' M_j \left(t_j - t_{j-1}\right)$ and $\sum' m_j \left(t_j - t_{j-1}\right)$ of f differ by less than $\frac{\epsilon}{4}$.

Moreover, the terms of the original upper and lower Riemann sums not found in the sums for the smaller interval amount to less than $\frac{\epsilon}{4}$.

In short, we have shown that for $c < \frac{\epsilon}{4M}$ and a suitable partition containing c,

$$\sum M_j \left(t_j - t_{j-1}\right) - \frac{\epsilon}{4} < \int_0^1 f(x)\, dx \le \sum m_j \left(t_j - t_{j-1}\right) + \frac{\epsilon}{4}$$

and

$$\sum' M_j \left(t_j - t_{j-1}\right) - \frac{\epsilon}{4} < \int_c^1 f(x)\, dx \le \sum' m_j \left(t_j - t_{j-1}\right) + \frac{\epsilon}{4}.$$

Moreover, we have also shown that

$$\left|\sum M_j \left(t_j - t_{j-1}\right) - \sum{}' M_j \left(t_j - t_{j-1}\right)\right| < \frac{\epsilon}{4}$$

and

$$\left|\sum m_j \left(t_j - t_{j-1}\right) - \sum{}' m_j \left(t_j - t_{j-1}\right)\right| < \frac{\epsilon}{4}$$

Combining these inequalities, we find that

$$\left|\int_0^1 f(x)\, dx - \int_c^1 f(x)\, dx\right| < \epsilon$$

If $0 < c < \frac{\epsilon}{4M} u$

(b) Let $f(x) = (-1)^n (n+1)$

for $\frac{1}{n+1} < x \leq \frac{1}{n}$, $n = 1,2,....$ Then if $\frac{1}{N+1} < c \leq \frac{1}{N}$ we have

$\int_c^1 f(x)\,dx = (-1)^N (N+1)\left(\frac{1}{N} - c\right) + \sum_{k=1}^{N-1} \frac{(-1)^k}{k}$.

Since $0 \leq \frac{1}{N} - c \leq \frac{1}{N} - \frac{1}{N+1} = \frac{1}{N(N+1)}$, the first term on the right hand side tends to zero as $c \downarrow 0$, while the sum approaches ln 2.

Hence this integral approaches a limit. However,

$$\int_c^1 |f(x)|\,dx = (N+1)\left(\frac{1}{N} - c\right) + \sum_{k=1}^{N-1} \frac{1}{k},$$

and in this case the first term on the right-hand side tends to zero as $c \downarrow 0$, while the sum tends to infinity.

13.2.3 Example : Suppose $f \in R$ on $[a, b]$ for every Suppose $b > a$, where a isfixed. Define $\int_a^\infty f(x)\,dx = \lim_{x\to\infty} \int_a^b f(x)\,dx$

If this limit exists (and is finite). In that case, we say that the integral on the left converges. If it also converges after f has been replaced by $|f|$, it is said to converge absolutely.

Assume that $f(x) \geq 0$ and that f decreases monotonically on $[1, \infty)$. Prove that

$\int_1^\infty f(x)\,dx$ converges if and only if $\sum_{n=1}^\infty f(n)$ converges. (This is the so-called " integral test" for convergence of series.)

Solution. Since both the series and the integral are increasing functions of their upper limits, it suffices to show that they are bounded together.

Define $f(x) = f(1)$ for $0 \leq x \leq 1$.

The upper Riemann sum for this partition is $\sum_{k=1}^{n-1} f(k)$ and

The lower Riemann sum is $\sum_{k=1}^{n} f(k)$.

Hence we have

$$\sum_{k=1}^{n} f(k) \leq \int_0^n f(x)\,dx = f(0) + \int_1^n f(x)\,dx \leq \sum_{k=1}^{n-1} f(k) = f(0) + \sum_{k=1}^{n-1} f(k).$$

This shows that

$-f(0) + \sum_{k=1}^{n} f(k) \leq \int_1^n f(x)\,dx \leq \sum_{k=1}^{n-1} f(k)$, And hence the sum and the integral converge or diverge together.

13.2.4 Example : Define $f(x) = \int_x^{x+1} \sin(t^2)\, dt$.

(a) Prove that $|f(x)| < 1/x$ if $x > 0$.

(b) Prove that $2xf(x) = \cos(x^2) - \cos[(x+1)^2] + r(x)$,

Where $|r(x)| < \frac{c}{x}$, and c is constant.

(c) Find the upper and lower limits of $xf(x)$ as $x \to \infty$.

(d) Does $\int_0^\infty \sin(t^2)\, dt$ converge?

Solution. (a) This inequality is obvious if $0 < x \leq 1$.

Hence we assume $x > 1$.

We observe that

$$f(x) < \frac{\cos(x^2)}{2x} - \frac{\cos[(x+1)^2]}{2(x+1)} + \frac{1}{2x} - \frac{1}{2(x+1)}$$

$$= \frac{1+\cos(x^2)}{2x} - \frac{1+\cos[(x+1)^2]}{2(x+1)}$$

$$\leq \frac{1+\cos(x^2)}{2x}$$

$$\leq \frac{1}{x}$$

A similar argument shows that

$$f(x) > \frac{\cos(x^2)}{2x} - \frac{\cos[(x+1)^2]}{2(x+1)} - \frac{1}{2x} + \frac{1}{2(x+1)}$$

$$= \frac{-1+\cos(x^2)}{2x} - \frac{-1+\cos[(x+1)^2]}{2(x+1)}$$

$$= \frac{-1+\cos(x^2)}{2x} + \frac{-1-\cos[(x+1)^2]}{2(x+1)}$$

$$\geq \frac{-1+\cos(x^2)}{2x}$$

$$\geq \frac{-1}{x}$$

(b) The expression just written for $f(x)$ shows that

$$2xf(x) = \cos(x^2) - \cos[(x+1)^2] + r(x),$$

Where

$$r(x) = \left(\frac{1}{x+1}\right)\cos[(x+1)^2] - \frac{x}{2}\int_{x^2}^{(x+1)^2} \frac{\cos u}{u^{3/2}}\,du.$$

If we integrate by parts again, we find that

$$\int_{x^2}^{(x+1)^2} \frac{\cos u}{u^{3/2}}\,du = \frac{\sin[(x+1)^2]}{(x+1)^3} - \frac{\sin[x^2]}{x^3} + \frac{3}{2}\int_{x^2}^{(x+1)^2} \frac{\sin u}{u^{5/2}}\,du.$$

We now observe that the absolute value of this last integral is at most

$$\frac{3}{2}\int_{x^2}^{\infty} \frac{1}{u^{5/2}}\,du = \left[-u^{3/2}\right]_{x^2}^{\infty} = x^{-3}$$

It then follows by collecting the terms that

$$|r(x)| < \frac{3}{x}.$$

(c) Since $r(x) \to 0$, the upper and lower limits of $xf(x)$ will be the corresponding limits of

$$\frac{\cos(x^2) - \cos[(x+1)^2]}{2} = \sin\left(x^2 + x + \frac{1}{2}\right)\sin\left(x + \frac{1}{2}\right).$$

We can write this last expression as $\sin s\,\sin\left(s^2 + \frac{1}{4}\right)$, where $s = x + \frac{1}{2}$.

We claim that the upper limit of this expression is 1 and the lower limit is -1.

Indeed, let $\epsilon > 0$ be given.

Choose n to be any positive integer larger than $\frac{2-\epsilon}{8\epsilon}$.

Then the interval $\left(\frac{1}{4} + \left(\left(2n + \frac{1}{2}\right)\pi - \epsilon\right)^2, \frac{1}{4} + \left(\left(2n + \frac{1}{2}\right)\pi + \epsilon\right)^2\right)$ is longer than $2\,\pi$,

And hence there exists a point $t \in \left(\left(2n + \frac{1}{2}\right)\pi - \epsilon, \left(2n + \frac{1}{2}\right)\pi + \epsilon\right)$

at which $\sin\left(t^2 + \frac{1}{4}\right) = 1$ and also a point u in the same interval at which

$$\sin\left(u^2 + \frac{1}{4}\right) = -1$$

But then $tf(t) > 1 - \epsilon$ and $uf(u) < -1 + \epsilon$

It follows that the upper limit is 1 and the lower limit is -1.

(This argument actually shows that the limit points of $xf(x)$ fill up the entire interval $[-1,1].$)

(d) The integral does converge.

We observe that for integers N we have

$$\int_0^N \sin(t^2)\, dt = \sum_{k=1}^{n} f(k)$$

$$= f(0) + \sum_{k=1}^{N} \frac{r(k)}{k} + \sum_{k=1}^{N} \frac{\cos(k^2) - \cos[(k+1)^2]}{k}$$

$$= f(0) + \sum_{k=1}^{N} \frac{r(k)}{k} + \left[\frac{\cos(1)}{2} - \frac{\cos[(N+1)^2]}{N}\right] + \sum_{k=2}^{N} \frac{\cos(k^2)}{k(k-1)}$$

The first sum on the right converges since $|r(k)| < \frac{3}{k}$, and the rest obviously converges.

Hence we will be finished if we show that

$$\lim_{x\to\infty} \int_{[x]}^{x} \sin(t^2)\, dt = 0,$$

Where $[x]$ is the largest such that $[x] \leq x < [x] + 1$.

But this is easily done using integration by parts.

The integral equals

$$\frac{\cos[x]^2}{2[x]} - \frac{\cos(x^2)}{x^2} - \int_{[x]^2}^{x^2} \frac{\cos u}{4u^{3/2}}\, du$$

And this expression obviously tends to zero as $x \to \infty$.

13.2.5 Example : Deal similarly with $f(x) = \int_x^{x+1} \sin(e^t)dt$.
Show that

$$e^x|f(x)| < 2$$

and that

$$e^x f(x) = \cos(e^x) - e^{-1}\cos(e^{x+1}) + r(x),$$

Where $|r(x)| < Ce^{-x}$ for some constant C.

Solution. The arguments are completely analogous to the preceding problem.

The substitution $u = e^t$ changes to $f(x)$ into

$$f(x) = \int_{e^x}^{e^{x+1}} \frac{\sin u}{u} du,$$

and then integration by parts yields

$$f(x) = \frac{\cos(e^x)}{e^x} - \frac{\cos(e^{x+1})}{e^{x+1}} - \int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2} du$$

from which it then follows that

$$-\frac{1-\cos(e^x)}{e^x} \le f(x) \le \frac{1+\cos(e^x)}{e^x}$$

We have the equality

$$e^x f(x) = \cos(e^x) - e^{-1}\cos(e^{x+1}) - e^x \int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2} du,$$

And one more integration by parts shows that

$$\left| e^x \int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2} du \right| < \frac{3}{e^x}$$

In this case $f(x)$ decreases so rapidly that there is no difficulty at all proving the converges of the integral.

13.2.6 Example : Suppose f is a real, continuously differentiable function on $[a, b]$, $f(a) = f(b) = 0$, and

$$\int_a^b f^2(x)dx = 1.$$

Prove that

$$\int_a^b x f(x) f'(x) dx = -\frac{1}{2}$$

and that

$$\int_a^b [f'(x)]^2 dx \,.\, \int_a^b x^2 f^2(x) dx \geq \frac{1}{4}.$$

Solution. To prove the first assertion we merely integrate by parts.

Taking $u = x$, $dv = f(x)f'(x)dx$,

So that $du = dx$ and $v = \frac{1}{2}f^2(x)$.

Since v vanishes at both end points, the result is

$$\int_a^b xf(x)\,f'(x)dx = -\frac{1}{2}\int_a^b f^2(x)dx = -\frac{1}{2}$$

The second inequality is an immediate consequence of the Schwarz inequality applied to the two functions $xf(x)$ and $f'(x)$.

Model Examination Questions

1. State and prove the fundamental theorem of calculus.
2. Suppose F and G are differentiable functions on $[a, b]$, $F' = f \in R$ and $G' = g \in R$ then show that
   $\int_a^b F(x)g(x)\,dx = F(b)g(b)F(a)G(a) - \int_a^b f(x)G(x)dx.$

13.3 SUMMARY:

This lesson uncovers the fundamental connection between integration and differentiation, revealing their inverse relationship. Through the lens of the Fundamental Theorem of Calculus, learners will explore key theorems and examples that illuminate this critical concept. Key Takeaways of this lesson are Integration and differentiation as inverse operations, The Fundamental Theorem of Calculus, Proofs and applications of selected integration theorems, and Examples with solutions to reinforce understanding.

13.4 TECHNICAL TERMS:

- Real valued function
- Uniformly Continuous
- Partition
- Upper Riemann Sum
- Lower Riemann Sum
- Constant function
- Upper Limit

- Lower Limit
- Converges absolutely
- Monotonically Decreasing
- Diverge

13.5 SELF ASSESSMENT QUESTIONS:

1. Let f be a real valued function on $[a, b]$ such that $f \in R[a, b]$. For $a \leq x \leq b$, put $F(x) = \int_a^x f(t)\, dt$. Then F is continuous on $[a, b]$; furthermore, if f is continuous at a point x_0 of $[a, b]$, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.
2. State and prove the fundamental theorem of calculus.
3. Suppose F and G are differentiable functions on $[a, b], F' = f \in R$ and $G' = g \in R$ then show that $\int_a^b F(x)g(x)\, dx = F(b)g(b)F(a)G(a) - \int_a^b f(x)G(x)dx$.
4. Suppose f is a real function on $[0,1]$ and $f \in R$ on $[c, 1]$ for every $c > 0$. Define $\int_0^1 f(x)\, dx = \lim_{c \to 0+} \int_c^1 f(x)\, dx$ if this limit exists (and is finite)

(a) If $f \in R$ on $[0,1]$ show that this definition of the integral agree.

(b) Construct a function f such that the above limit exists, although it fails to exist with $|f|$ in place of f.

13.6 SUGGESTED READINGS:

1. Principles of Mathematical Analysis, Third Edition, Mc Graw-Hill International Editions Walter Rudin.

2. Mathematical Analysis by Tom M. Apostol, Narosa Publishing House, 2nd Edition, 1985.

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LESSON-14

INTEGRATION OF VECTOR-VALUED FUNCTIONS AND RECTIFIABLE CURVES

OBJECTIVES:

The objective of the lesson is to help the learners to understand

- To define and compute integrals of vector-valued functions.
- To understand and apply properties of rectifiable curves.

STRUCTURE:

14.0 INTRODUCTION:

In the lesson, we define vector valued function on $[a, b]$ into R_k, and proved some properties of vector valued function. Also defined the rectifiable curve and derived the formulae for length of the rectifiable curve on $[a, b]$

We define the integral of a vector valued function as the integral of each component. This definition holds for both definite and indefinite integrals.

14.1 : DEFINITION:

Integration of vector valued functions let $f_1, f_2, \dots, f_k$ be real

valued functions on $[a, b]$ and let $\vec{f} = (f_1, f_2, \dots, f_k)$.

Be real valued functions $[a, b]$ and let $f = (f_1, f_2, \dots, f_k)$ be the corresponding vector valued function of $[a, b]$ into $\mathbb{R}^k$. Let α be monotonically increasing function on $[a, b]$, we say that $f \in R(\alpha)$ on $[a, b]$ if $f_j \in R(\alpha)$ on $[a, b]$ for, $1 \le i \le k$. If this is the case, we define

$$\int_a^b \vec{f}\, d\alpha = \left(\int_a^b f_1\, d\alpha, \dots, \int_a^b f_k\, d\alpha \right)$$

14.1.1 Theorem: If $\vec{f}, \vec{g} \in R$ on $[a, b]$, then

(i) $\vec{f} + \vec{g}$

(ii) $cf \in R$ on $[a, b]$ for every constant c and

$$\int_a^b (\vec{f} + \vec{g}) d\alpha = \int_a^b \vec{f}_1 d\alpha + \int_a^b \vec{g}_1 d\alpha$$

and

$$\int_a^b c\vec{f} d\alpha = c \int_a^b \vec{f} d\alpha$$

Proof: Suppose $\vec{f} = (f_1, f_2, \ldots, f_k)$ and $\vec{g} = (g_1, g_2, \ldots, g_k)$ are vector valued functions of $[a, b]$ into $\mathbb{R}^k$ and $\vec{f}, \vec{g} \in R(\alpha)$ on $[a, b]$.

Then, $f_i \in R(\alpha)$ on $[a, b]$ for $1 \leq i \leq k$ and $g_i \in R(\alpha)$ on $[a, b]$ for $1 \leq i \leq k$.

By theorem 13.1.2, $f_i + g_i \in R(\alpha)$ on $[a, b]$ for $1 \leq i \leq k$ and

$$\int_a^b (f_i + g_i) d\alpha = \int_a^b f_i d\alpha + \int_a^b g_i d\alpha \text{ for } 1 \leq i \leq k$$

Since $\vec{f} + \vec{g} = (f_1 + g_1, f_2 + g_2, \ldots, f_k + g_k)$ and

$f_i + g_i \in R(\alpha)$ on $[a, b]$ for $1 \leq i \leq k$, we have $f_i + g_i \in R(\alpha)$ on $[a, b]$.

$$\int_a^b (\vec{f} + \vec{g}) d\alpha = \left[\int_a^b (f_1 + g_1) d\alpha, \int_a^b (f_2 + g_2) d\alpha, \ldots, \int_a^b (f_k + g_k) d\alpha \right]$$

Thus we have proved (i)

Let c be any constant

By Theorem 12.1.12, $cf_i \in R(\alpha)$ on $[a, b]$ and

$$\int_a^b cf_i d\alpha = c \int_a^b f_i d\alpha \text{ for } 1 \leq i \leq k$$

Since $c\vec{f} - (cf_1, cf_2, \ldots, cf_k)$ we have $c\vec{f} \in R(\alpha)$ on $[a, b]$ and

$$\int_a^b c\vec{f} d\alpha = \left[\int_a^b cf_1 d\alpha, \int_a^b cf_2 d\alpha, \ldots, \int_a^b cf_i d\alpha \right]$$

Thus we have proved (ii)

Similarly we prove the following Theorem by using Theorem 12.1.4, Theorem 12.1.6 and Theorem 12.1.7.

14.1.2 Theorem: Let $\boldsymbol{f}$ be a vector-valued function on $[\boldsymbol{a}, \boldsymbol{b}]$ and $\mathbb{R}^{\boldsymbol{k}}$

(i) If $\vec{\boldsymbol{f}} \in \boldsymbol{R}(\boldsymbol{\alpha})$ on $[\boldsymbol{a}, \boldsymbol{b}]$ and if $\boldsymbol{a} < c < b$, then on $[\boldsymbol{a}, \boldsymbol{c}]$ and $\boldsymbol{f} \in \boldsymbol{R}(\boldsymbol{\alpha})$ on $[\boldsymbol{c}, \boldsymbol{b})$ and

$$\int_a^b \vec{f} d\alpha = \int_a^b \vec{f} d\alpha + \int_b^c \vec{f} d\alpha$$

(ii) If $\vec{f} \in R(\alpha_1)$ and $\vec{f} \in R(\alpha_2)$ on $[a, b]$ then $\vec{f} \in R(\alpha_1 + \alpha_2)$ and

$$\int_u^b \vec{f} d(\alpha_1 + \alpha_2) = \int_u^b \vec{f} d\alpha_1 + \int_u^b \vec{f} d\alpha_2$$

(iii) If $\vec{\boldsymbol{f}} \in \boldsymbol{R}(\boldsymbol{\alpha})$ on $[\boldsymbol{a}, \boldsymbol{b}]$ and $\boldsymbol{c}$ is a positive constant, then $\vec{\boldsymbol{f}} \in \boldsymbol{R}(\boldsymbol{c\alpha})$ and

$$\int_a^b \vec{f} d(c\alpha) - \int_a^b \vec{f} d(\alpha)$$

Theorem 12.1.1 is also true for vector-valued functions.

14.1.3 : Theorem: If $\vec{f}$ and $\vec{F}$ map $[a, b]$ into $\mathbb{R}^k$, if $f \in R(\alpha)$ on $[a, b]$ and $\vec{F} = \vec{f}$, then

$$\int_a^b \vec{f}(t) dt = \vec{F}(b) = \vec{f}(a).$$

Proof: Suppose $\vec{f} = (f_1, f_2, \dots, f_k)$ and map $[a, b]$ into $\mathbb{R}^k$ and $\vec{f} \in R$ and $\vec{F'} = \vec{f}$ then $f_i \in R$ on $[u, b]$ and $F_1'(a) = \vec{f_i}$ for $1 \le i \le k$

By known theorem,

$$\int_a^b \vec{f}(x) dx - \vec{F'}(b) - \vec{F}(a), 1 \le i \le k,$$

Therefore,

$$\int_a^b \vec{f}(x) dx = \left[\int_a^b f_1(x) dx, \int_a^b f_2(x) dx, \dots, \int_a^b f_k(x) dx\right]$$

$$= F_1(b) - F_1(a), F_2(b) - F_2(a), \dots, F_k(b) - F_k(a)$$

$$= \vec{F'}(b) - \vec{F}(a)$$

Therefore,

$$\int_a^b \vec{f}(x) dx = \vec{F'}(b) - \vec{F}(a).$$

Since x^2 is a continuous function of x, by a known theorem, the square root function is continuous on $[0, M]$ for every positive real number M.

Since $|\vec{f}| = (f_1^2 + f_2^2 + \dots + f_k^2)^{\frac{1}{2}}$, by Theorem 11.1.6 we have $|f| \in R(\alpha)$ on $[a, b]$.

Now, we will show that

$$\left|\int_a^b \vec{f}d\alpha\right| \leq \int_a^b |\vec{f}|d\alpha.$$

Put

$$y_j = \int_a^b f_j d\alpha \ for\ 1 \leq i \leq k,$$

and write $y = (y_1, y_2, y_3, \dots, y_k)$

Then we have,

$$y = \int_a^b \vec{f}d\alpha$$

and

$$|y^2| = \sum_{j=1}^{k} y_1^2 = \sum_{j=1}^{k} y_j \int_a^b f_j = \int_a^b \left(\sum_{j=1}^{k} y_j f_j\right)d\alpha$$

By the Schwarz inequality,

$$\sum_{j=1}^{k} y_j f_j(t) \leq |y||f(t)|, for\ all\ t \in [a,b]$$

By Theorem 12.1.3

$$|\vec{y}|^2 \leq |\vec{y}| \int_a^b |\vec{f}|\,d\alpha \cdots\cdots\cdots\cdots (1)$$

If $y - 0$, then trivially

$$\left|\int_a^b \vec{f}d\alpha\right| \leq \int_a^b |\vec{f}|d\alpha$$

If $|\vec{y}|$=0 then divide (1) by $|\vec{y}|$ on both sides. Then we have

$$|\vec{y}| \leq \int_a^b |\vec{f}|d\alpha \left|\int_a^b \vec{f}d\alpha\right| \leq \int_a^b |\vec{f}|d\alpha$$

14.2 : RECTIFIABLE CURVES:

14.2.1 Definition: A continuous mapping γ of an interval $[a, b]$ into $\mathbb{R}^k$ is called a curve in $\mathbb{R}^k$. In this case we sometimes say that γ is a curve on $[a, b]$.

1. If γ is one -to - one, γ is called an arc.
2. If $\gamma(a) = \gamma(b)$, γ is said to be a closed curve. We associate to each partition

$P = \{x_0, x_1, x_2 \ldots, x_n\}$ of $[a, b]$ and to each curve γ on $[a, b]$ the number.

$$\wedge (P, \gamma) = \sum_{i=1}^{n} |\gamma(x_1) - \gamma(x_{i-1})|$$

The i^{th} term in this sum is the distance (in $\mathbb{R}^k$) between the points $\gamma(x_{i-1})$ and $\gamma(x_i)$ hence $\wedge (p, \gamma)$ is the length of a polygonal path with vertices at $\gamma(x_0), \gamma(x_1), \ldots, \gamma(x_n)$ in this order. This polygon approaches the range of γ if $\|p\| \to 0$. Hence the following definition is reasonable.

14.2.2 : Definition: Let γ be a curve on $[a, b]$. We define the length of γ, defined by $\wedge (\gamma)$, as $\wedge (\gamma) = \sup\{\wedge (p, \gamma) / P \text{ is a partition of } [a, b]\}$

We say that r is rectifiable, if $\wedge (\gamma)$ is finite.

In the case of continuously differentiable curves, i.e. for curves γ whose derivative γ' is continuous. $\wedge (\gamma)$ is given by a Riemann integral.

14.2.3 : Theorem : If γ' is continuous on $[a, b]$ then γ is rectifiable and $\wedge (\gamma) = \int_a^b |\gamma'(t)| dt$.

Proof: Suppose $\wedge (\gamma) = \int_a^b \|\gamma'\| \, dt$ is continuously differentiable on $[a, b]$,

let $P = \{x_0, x_1, x_2 \ldots, x_n\}$ be any partition of $[a, b]$.

Consider,

$$|\gamma(x_i) - \gamma(x_{i-1})| = \left| \int_{x_{i-1}}^{x_i} \gamma'(t) \, dt \right|$$

$$\leq \int_{x_{i-1}}^{x_i} |\gamma'(t)| \, dt$$

for $1 \leq i \leq k$.

(By Theorem 12.1.18) This implies that

$$\wedge (p, \gamma) = \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})|$$

$$\leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |\gamma'(t)| \, dt$$

$$= \int_a^b |\gamma'(t)| \, dt$$

So for any partition P of $[a, b]$,

$$\wedge (p, \gamma) \leq \int_a^b |\gamma'(t)| \, dt$$

Consequently,

$$\wedge(\gamma) \leq \int_a^b |\gamma'(t)|\, dt \cdots\cdots\cdots\cdots\cdots (1).$$

Let $\varepsilon > 0$. Write $\epsilon_1 = \frac{\epsilon}{2((b-a)+1)}$

Since γ is continuously differentiable on $[a, b]$. γ' is continuous on $[a, b]$. Since $[a, b]$ is compact and γ' is continuous on $[a, b]$, γ' is uniformly continuous on $[a, b]$.Then there exist $\delta > 0$ such that $|\gamma'(s) - \gamma'(t)| \leq \epsilon_1$.

Consider,

$$\int_a^b |\gamma'(t)|\, dt = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |\gamma'(t)|\, dt$$

$$\leq \sum_{i=1}^{n} |\gamma(x_i) - \gamma(t)| + 2\epsilon_1 \sum_{i=1}^{n} \Delta x_i$$

$$= \wedge(p, \gamma) + 2\epsilon_1(b - a)$$

$$\leq \wedge(\gamma) + 2\epsilon_1(b - a)$$

$$= \wedge(\gamma) + \frac{2\epsilon(b-a)}{2((b-a)+1)}$$

$$< \wedge(\gamma) + \varepsilon$$

Therefore,

$$\int_a^b |\gamma'(t)|\, dt \leq \wedge(\gamma) + \epsilon$$

Since $\epsilon > 0$ is arbitrary,

$$\int_a^b |\gamma'(t)|\, dt \leq \wedge(\gamma) \cdots\cdots\cdots\cdots\cdots\cdots (3)$$

From (1) and (3),

$$\int_a^b |\gamma'(t)|\, dt \leq \wedge(\gamma).$$

14.3 SOME MORE EXAMPLES WITH SOLUTIONS:

14.3.1 Example: For $1 < s < \infty$, define $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$

Prove that

(a) $\zeta(s) = s\int_1^{\infty} \frac{[x]}{x^{s+1}} dx$

and that

(b) $\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{x-[x]}{x^{s+1}}dx,$

Where $[x]$ denotes the greatest integer $\leq x$.

Prove that the integral in **(b)** converges for all $x > 0$.

Hint: To prove **(a)** compute the difference between the integral over $[1, N]$ and the Nth partial sum of the series that defines $\zeta(s)$.

Solution: (a) Ignoring the author's advice, we note that

$$s\int_1^\infty \frac{[x]}{x^{s+1}}dx = s\sum_{n=1}^\infty n\int_n^{n+1} s\int_1^\infty \frac{1}{x^{s+1}}dx$$

$$= \sum_{n=1}^\infty n\left[\frac{1}{n^s} - \frac{1}{(n+1)^s}\right]$$

$$= 1\left[\frac{1}{1^s} - \frac{1}{2^s}\right] + 2\left[\frac{1}{2^s} - \frac{1}{3^s}\right] + \cdots$$

$$= \sum_{n=1}^\infty \frac{1}{n^s}$$

$$= \zeta(s).$$

(b) This result is trivial consequence of **(a)** and the identity

$$\frac{s}{s-1} = \int_1^\infty \frac{x}{x^{s+1}}dx.$$

14.3.2 Example: Suppose α increases monotonically on $[a, b]$, g is continuous, and $g(x) = G'(x)$ for $a \leq x \leq b$. Prove that

$$\int_a^b \alpha(x)g(x)\,dx = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G\,d\alpha.$$

Hint: Take g real, without of generality.

Given $P = \{x_0, x_1, \ldots, x_n\}$, choose $t_i \in (x_{i-1}, x_i)$ so that $g(t_i)\Delta x_i = G(x_i) - G(x_{i-1})$.

Show that $\sum_{i=1}^n \alpha(x_i)\,g(t_i)\Delta x_i = G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^n G(x_{i-1})\Delta\alpha_i$.

Solution: The identity just given is a trivial consequence of Abel's method of rearranging the sums:

$$\sum_{i=1}^n \alpha(x_i)g(t_i)\Delta x_i = \sum_{i=1}^n \alpha(x_i)\left(G(x_i) - G(x_{i-1})\right)$$

$$= G(x_n)\alpha(x_n) - G(x_0)\alpha(x_0) - \sum_{i=1}^n (x_{i-1})\left(\alpha(x_i) - \alpha(x_{i-1})\right).$$

Now the fact that $G(x)$ is continuous and α is non-decreasing means that the right-hand side can be made arbitrarily close to

$$G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G\, d\alpha,$$

whenever the partition is sufficiently fine.

It does not follow immediately that the function $\alpha(x)g(x)$ is integrable on $[a,b]$.

However, since α is non decreasing, its only discontinuities are jumps, and for any $\epsilon > 0$ there can be only a finite number of jumps larger than ϵ.

These can be enclosed in a finite number of open intervals of arbitrary small length.

We can then argue, that any partition that is sufficiently fine will have upper and lower Riemann sums that differ by less than ϵ.

Hence $\alpha(x)g(x)$ is integrable, and its integral is given by the stated relation.

14.3.3 Example: Let $\gamma_1, \gamma_2, \gamma_3$ be curves in the complex plane defined on $[0, 2\pi]$ by $\gamma_1(t) = e^{it}$, $\gamma_2(t) = e^{2it}$, $\gamma_3(t) = e^{2\pi i t sin\,(1/t)}$.

Show that these curves have the same range, that γ_1 and γ_2 are rectifiable, that the length of γ_1 is 2π, that the length of γ_2 is 4π, and that γ_3 is not rectifiable.

Solution: Since e^{it} has period 2π

It is obvious that γ_1 and γ_2 have the same range, namely the set of all complex numbers of absolute value 1.

To show that this is also the range of γ_3

We need to show that the mapping $t \to 2\pi t sin\,(1/t), 0 \leq t \leq 2pi$, covers an interval of length 2π

i.e., that the mapping $t \to tsin\,(1/t). 0 < t < 2\pi$ covers an interval of length 1. (We naturally take the value to be zero when $t = 0$)

Since this range is connected, it suffices to find two points a and b in the range with $a - b > 1$.

We choose those points to be $a = \frac{3}{\pi}$ (the image of $t = \frac{6}{\pi}$)

and $b = -\frac{2}{3\pi}$ (the image of $t = \frac{2}{3\pi}$)

We have $a - b = \frac{11}{3\pi} > 1$.

The rectification of γ_1 and γ_2 is straight forward:

$$l(\gamma_1) = \int_0^{2\pi} |\gamma_1'(t)|\, dt = 2\pi,$$

$$l(\gamma_2) = \int_0^{2\pi} |\gamma_2'(t)|\, dt = 4\pi$$

To show that γ_3 is not rectifiable, we observe that its length would be

$$\int_0^{2\pi} \left| sin\,(1/t) - \frac{1}{t}\cos(1/t) \right| dt \geq \int_0^{2\pi} \left| \frac{\cos(1/t)}{t} \right| dt - 2\pi.$$

By making the substitution $u = \frac{1}{t}$ in the last integral we get

$$\int_{\frac{1}{2\pi}}^{\infty} \left|\frac{\cos u}{u}\right| du$$

But we already know that this integral diverges, since

$$\sum_{n=1}^{\infty} \int_{2n\pi}^{\left(2n+\frac{1}{2}\right)\pi} \frac{\cos u}{u} \geq du \sum_{n=1}^{\infty} \frac{1}{\left(2n+\frac{1}{2}\right)\pi} = \infty.$$

14.3.4 Example: Let γ_1 be a curve in R^k defined on $[a,b]$; let φ be a continuous one-one mapping of $[c,d]$ onto $[a,b]$ such that $\varphi(c) = a$, and define $\gamma_2(x) = \gamma_1(\varphi(x))$. Prove that γ_2 is an arc, a closed curve, or a rectifiable curve if and only if the same is true of γ_1. Prove that γ_1 and γ_2 have the same length.

Solution: We know that φ has a continuous 1-1 inverse φ.

And that the composition of one-to-one functions is one-to-one.

Hence, since $\gamma_1(x) = \gamma_2(\varphi(x))$,

We see that γ_1 and γ_2 are both arcs (one-to-one) if either is.

Since necessarily $\varphi(d) = b$, we see that $\gamma_1(a) = \gamma_1(b)$ if and only if $\gamma_2(c) = \gamma_2(d)$.

Hence both are closed curves if either is.

Finally, since φ and φ establish a one-to-one correspondence between partitions $\{s_i\}$ of $[a,b]$ and $\{t_i\}$ of $[c,d]$ such that $\sum|\gamma_1(s_i) - \gamma_1(s_{i-1})| = \sum|\gamma_2(t_i) - \gamma_2(t_{i-1})|$.

It follows that the two curves have the same length.

14.3.5 Example: Evaluate

$$\int (\sin t)\,\hat{\imath} \;|\; 2t\,\hat{\jmath} \quad 8t^3\,\hat{k}\,dt.$$

Solution: Just take the integral of each component

$$\int (\sin t)\,\hat{\imath}\,dt + \int 2t\,\hat{\jmath}\,dt - \int 8t^3\,\hat{k}\,dt\,.$$

$$= (-\cos t + c_1)\,\hat{\imath} + (t^2 + c_2)\,\hat{\jmath} + (2t^4 + c_3)\hat{k}.$$

14.3.6 Note: We have introduce three different constants, one for each component.

14.3.7 Example. Suppose $f \geq 0, f$ is continuous on $[a,b]$ and $\int_a^b f\,dx = 0$ then prove that $f(x) = 0$, for all $x \in [a,b]$.

Solution. Let P be a partition of $[a,b]$

Since f is continuous on $[a,b]$

Then $f \in R$ on $[a,b]$

Since f is continuous on $[a,b]$ then f attains its maximum, so $M_i = f(t_i)$ for some $t_i \in [x_{i-1}, x_i]$

Now for any partition P, $U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$

$$= \sum_{i=1}^{n} f(t_i)\Delta x_i$$

Since $Inf\ U(P,f) = \int_a^{\bar{b}} f\,dx$

$$= \int_a^b f\,dx$$

$= 0$

$\Rightarrow \inf U(p,f) = 0$

$$\Rightarrow \sum_{i=1}^{n} f(t_i)\Delta x_i = 0$$

$\Rightarrow f(t_i) = 0$ for $t_i \in [a,b]$

$\Rightarrow f(x) = 0$ for all $x \in [a,b]$.

14.3.8 Example. If $f \in R$ on $[a,b]$ then $|f| \in R$ on $[a,b]$. Show that the converse need not be true.

Solution. Define $f: [a,b] \to R$ by

$$f(x) = \begin{cases} 1\ if\ x\ is\ rational\ number \\ -1\ if\ x\ is\ irrational \end{cases}$$

$|f(x)| = 1$, for all $x \in [a,b]$

$|f|$ is constant function on $[a,b]$

$|f|$ is continuous on $[a,b]$

$|f| \in R$ on $[a,b]$ and $\int_a^b f\,dx = 1$

Since $f \notin R$ on $[a,b]$

Therefore $|f| \in R$ on $[a,b]$ but $f \notin R$ on $[a,b]$.

14.3.9 Example. Suppose α is increasing on $f \notin R$ on $[a,b], a \le x_0 \le b$, α is continuous at x_0, $f(x_0) = 1$ and $f(x) = 0$ if $x \ne x_0$ then prove that $f \in R(\alpha)$ and $\int_a^b f\,d\alpha = 0$.

Solution. Define f as $f(x) = \begin{cases} 0, \forall x \ne x_0 \\ 1, if x = x_0 \end{cases}$

Therefore we have $f(x_0^+) = f(x_0^-) = 0$

But $f(x_0) = 1$

$\Rightarrow f(x_0^+) = f(x_0^-) \ne f(x_0)$

$\to f$ is discontinuous at x_0

Therefore f has only one discontinuity in $[a,b]$

Given that α is continuous at x_0

By Known theorem $f \in R(\alpha)$ on $[a, b]$

Let P be a partition of $[a, b]$

We have $m_i = 0, 1 \leq i \leq n$

$$\Rightarrow \sum_{i=1}^{n} m_i \Delta\alpha_i = 0$$

$\Rightarrow L(p, f, \alpha) = 0$ for ant partition P of $[a, b]$

$\Rightarrow Sup L(p, f, \alpha) = 0$

$$\Rightarrow \int_a^b f \, d\alpha = 0$$

Therefore $\int_a^b f \, d\alpha = 0.$

14.4 SUMMARY:

This lesson introduces the concept of integrals of vector-valued functions, exploring their definition, computation, and properties. Learners will also delve into the concept of rectifiable curves, understanding their properties and applications. The components of this lesson is to Introduce integrals of vector-valued functions, Definition and computation of integrals, Theorems with proofs, Rectifiable curves: definition, properties, and applications, and Examples with solutions to illustrate key concepts.

14.5 TECHNICAL TERMS:

- Arcs
- Closed curve
- Compact
- Complex plane
- Constant function
- Continuity
- Derivative
- Discontinuity
- Functions
- Integral of each component
- Monotonically
- Partition
- Rectifiable curve
- Series
- Vector

14.6 SELF ASSESSMENT QUESTIONS

1. Define Integration of vector valued functions.

2. Define arc.

3. Define closed curve.

4. If $\vec{f}, \vec{g} \in R$ on $[a, b]$, then

(i) $\vec{f} + \vec{g}$

(ii) $cf \in R$ on $[a, b]$ for every constant c and

$$\int_a^b (\vec{f} \mid \vec{g})\,d\alpha - \int_a^b \vec{f}_1\,d\alpha + \int_a^b \vec{g}_1\,d\alpha$$

and

$$\int_a^b c\vec{f}\,d\alpha = c\int_a^b \vec{f}\,d\alpha$$

5. If γ' is continuous on $[a, b]$ then γ is rectifiable and $\wedge(\gamma) = \int_a^b |\gamma'(t)|\,dt$.

14.7 SUGGESTED READINGS:

1. Principles of Mathematical Analysis, Third Edition, Mc Graw-Hill International Editions Walter Rudin.

2. Mathematical Analysis by Tom M. Apostol, Narosa Publishing House, 2nd Edition, 1985.

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